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RESEARCH REPORT No. HT-7

Dynamics of Nonlinear Stochastic Systems

ROBERT H. KRAICHNAN

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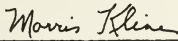
Research Report No. HT-7

DYNAMICS OF NONLINEAR STOCHASTIC SYSTEMS

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July, 1960

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Table of Contents

	<u>Page</u>
Abstract	2
1. Introduction	3
2. The random oscillator	6
3. Collective representation for a set of oscillators	11
4. Formulation of model problems	16
5. The random coupling model	24
6. Inadmissible higher approximations	28
7. Second stochastic model	32
8. Driven random oscillator	39
9. Non-Gaussian frequency distribution	47
10. Particle in a random potential	49
11. Turbulence dynamics	61
Acknowledgements	73
Appendix A	74
Footnotes	76
Figures	

A method for treating nonlinear stochastic systems is described which it is hoped will be useful in both the quantum-mechanical many-body problem and the theory of turbulence. In this method the true problem is replaced by models that lead to closed equations for correlation functions and averaged Green's functions. The model solutions are exact descriptions of possible dynamical systems, and as a result they display certain consistency properties. For example, spectral components of Green's functions which must be positive-definite in the true problem automatically are so for the models. The models involve a new stochastic element: Random couplings are introduced among an infinite collection of similar systems, the true problem corresponding to the limit where these couplings vanish. The method is first applied to a linear oscillator with random frequency parameter. The mean impulse-response function of the oscillator is obtained explicitly for two successive models. The results suggest the existence of a sequence of model solutions which converges rapidly to the exact solution of the true problem. Applications then are made to the Schrödinger equation of a particle in a random potential and to Burgers' analog for turbulence dynamics. For both problems, closed model equations are obtained which determine the average Green's function, the amplitude of the mean field, and the covariance of the fluctuating field. The model solutions can be expressed as sums of infinite classes of terms from the formal perturbation expansions of the solutions to the true problems. It is suggested that correspondence to stochastic models may be a useful criterion to help judge the validity of partial summations of perturbation series.

1. INTRODUCTION

This paper is intended to introduce a method for treating certain problems where the dynamical equations are nonlinear in stochastic quantities. The quantum-mechanical many-body problem^{1,2} and the theory of turbulence^{3,4} are two fields of current interest where it is hoped that the method will prove useful. In such problems, there arise from the dynamical equations an infinite hierarchy of coupled equations which relate given ensemble averages to successively more complicated ones. An equivalent statement is that the prediction of a given average over a finite time requires the initial knowledge of an infinite number of averages. This situation, which commonly is called the closure problem, arises even when the nonlinear stochastic terms are linear in the dynamic variables. An example is linear wave propagation in a medium with random refractive index fluctuations.⁵ Here the equation for the ensemble-averaged wave amplitude forms the base of an hierarchy involving successively higher cross-moments of the joint distribution of index and amplitude fluctuations.

A formal solution to the dynamical equations of any of the problems mentioned above may be obtained by treating the nonlinear terms as a perturbation and expanding by iteration.^{6,3} One may then approximate statistical quantities by either truncating this expansion or summing tractable classes of terms to all orders. Another (and related) approach is to discard all cumulants of the statistical distribution above a certain order. Then all averages are expressible in terms of averages of this order and below, thereby providing a closure of the hierarchy of coupled statistical equations.^{2,7}

In the method to be presented here, the true problem is replaced by models that lead, without approximation, to closed equations for correlation functions and averaged Green's functions. The model solutions are exact descriptions of

possible dynamical systems, and consequently they have certain consistency properties which can be lacking in the approximation schemes mentioned above. For example, spectral components of Green's functions which must be positive-definite in the true problem automatically are so in the models. A related property is that covariances satisfy certain realizability inequalities.

The models are constructed by introducing dynamical couplings among an infinite collection of similar systems, the true problem corresponding to the limit in which these couplings vanish. The coupling coefficients change randomly from one individual system in the collection to another. Thus they constitute a new stochastic element not present in the true problem. The models are most easily formulated in terms of a collective representation in which the variables are linear combinations of those of all the individual systems.

The closed statistical equations which characterize the models are obtained by averaging over an ensemble of realizations of the collection of coupled systems. When iteration expansions are generated for the averages of basic interest, it is found, using the collective representation, that the random couplings result in the cancellation of large classes of terms of all orders. The remaining terms are identical with corresponding ones in the expansion for the true problem (zero couplings). Although still of all orders, they have a sufficiently simple structure that their sum represents the exact solution of closed integral equations.

The method of stochastic models is introduced in the present paper by application to a linear oscillator whose frequency parameter is Gaussianly distributed over an ensemble. This system has the virtue that it can be solved exactly. Furthermore, it displays great sensitivity to inadequacies in approximation schemes. Neither truncation of the perturbation series nor the cumulant-

discard approach yields admissible approximations (Section 2). The collective representation and the general model are formulated in Sections 3 and 4. Explicit solutions for the average impulse-response function of the oscillator then are obtained for two particular models (Sections 5 and 7). They suggest the existence of a sequence of model solutions which converges rapidly to the exact solution for the true problem. In Section 8, model equations are obtained for the mean and covariance of the amplitude of the oscillator when driven by random forces. The generalization to non-Gaussian frequency distributions is described in Section 9.

In Section 6, approximations for the average response function are examined which represent infinite classes of terms in the perturbation expansion for the true problem, but which do not correspond to possible stochastic models. Although they are very plausible in terms of a diagrammatic representation of the perturbation series, these approximations have pathological characteristics. This suggests that correspondence to stochastic models may be a useful criterion to help judge the validity of partial summations of perturbation series in analogous situations.

In Sections 10 and 11, stochastic models are formulated for two problems of more physical interest: the Schrödinger equation of a particle in a random potential and Burgers' analog to turbulence dynamics. For both problems, closed integral equations are obtained which determine the average Green's function, the amplitude of the mean field, and the covariance of the fluctuating field. The models for these systems have an intimate formal relation to those for the random oscillator. In fact, the random potential problem is homologous to the oscillator problem in the sense that the coupling coefficients characterizing

corresponding models are identical in the two cases. Many results for the random potential problem can be obtained by inspection from the oscillator results.

A comparison of the model equations for the random potential and turbulence problems illustrates the similarities and differences involved when the present method is applied to systems which are, respectively, linear and nonlinear in the dynamic variables.

In a paper to follow, stochastic models are formulated for classical and quantized nonlinear oscillators. Then the many-boson problem with interparticle forces is treated. This problem is homologous to the quantized nonlinear oscillator in the same way as the random potential problem is to the classical random linear oscillator. Particular attention is given to thermal equilibrium. The Einstein-Bose distribution law is derived by requiring equilibrium under arbitrary infinitesimal changes in the coupling among systems in a collection, without assuming the grand canonical or other such distribution.

2. THE RANDOM OSCILLATOR

Let the amplitude $q(t)$ of a linear oscillator satisfy

$$dq(t)/dt + ibq(t) = 0, \quad (2.1)$$

where b is a real, time-independent parameter which is statistically distributed over an infinite ensemble of realizations of the oscillator. We shall be interested in determining the function $G(t) = \langle G_{[\]}(t) \rangle$, where $\langle \rangle$ denotes ensemble average and $G_{[\]}(t)$ is the response function defined for $-\infty < t < \infty$ by⁸

$$dG_{[\]}(t)/dt = -ibG_{[\]}(t), \quad G_{[\]}(0) = 1. \quad (2.2)$$

We have, immediately, $G(t) = \langle \exp(-ibt) \rangle = \int_{-\infty}^{\infty} \exp(-ibt) P(b) db$, where

$P(b)$ is the normalized probability density for b . Hence,

$$\tilde{G}(\omega) = P(\omega) , \quad (2.3)$$

where

$$\tilde{G}(\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} G(t) \exp(i\omega t) dt .$$

Since $P(b) \geq 0$, $\tilde{G}(\omega)$ must satisfy the realizability condition

$$\tilde{G}(\omega) = |\tilde{G}(\omega)| . \quad (2.4)$$

A particular consequence of (2.4) is

$$|G(t)| \leq G(0) = 1 , \quad (2.5)$$

which also follows from the fact that $|q(t)|$ is a constant of motion in each realization of the oscillator.

Now suppose that $P(b)$ is not known in closed form, but instead is specified by the infinite set of moments $\langle b \rangle$, $\langle b^2 \rangle$, $\langle b^3 \rangle$, ... Then, by integrating (2.2) from 0 to t , iterating, and averaging, we may generate the formal solution

$$G(t) = 1 + \sum_{n=1}^{\infty} (-i)^n \langle b^n \rangle t^n / n! . \quad (2.6)$$

Eq. (2.6) corresponds precisely to the perturbation series for the ^{averaged} Green's function in certain statistical field physics problems. Let us explore its validity for the present problem by taking the example of Gaussian $P(b)$. Then

we have

$$\tilde{G}(\omega) = (2\pi \langle b^2 \rangle)^{-1/2} \exp(-\frac{1}{2} \omega^2 / \langle b^2 \rangle) , \quad (2.7)$$

$$G(t) = \exp(-\frac{1}{2} \langle b^2 \rangle t^2) \quad (2.8)$$

But let us suppose that we do not know this closed form and instead are given the moment values

$$\langle b^n \rangle = 0 \quad (n \text{ odd}), \quad \langle b^{2n} \rangle = (\frac{1}{2} \langle b^2 \rangle)^n n! / n! . \quad (2.9)$$

By (2.6) we have

$$G(t) = 1 + \sum_{n=1}^{\infty} (-\frac{1}{2} \langle b^2 \rangle t^2)^n / n! , \quad (2.10)$$

which, of course, is the power series expansion of (2.8) .

The following observations may be made concerning (2.10). First, it is absolutely convergent for all t . Second, for $t > 2/\langle b^2 \rangle^{1/2}$ the convergence rapidly becomes very poor so that very many terms must be taken to obtain a good approximation. Third, if the series is truncated after any finite number of terms, we have $G(t) \rightarrow \infty$, $t \rightarrow \infty$, in violation of the basic realizability condition (2.5). Thus, at no finite stage of the iteration treatment do we obtain an approximation with uniform validity for all t , and, in particular, at no stage does the spectral density $\tilde{G}(\omega)$ exist.

Let us next apply a second approximation scheme which has been widely used in statistical field physics. From (2.2) we may obtain the infinite set

of coupled equations

$$\begin{aligned}
 dG(t)/dt &= -1 \langle bG_{[]} (t) \rangle & G(0) &= 1, \\
 d \langle bG_{[]} (t) \rangle /dt &= -1 \langle b^2 G_{[]} (t) \rangle & \langle bG_{[]} (0) \rangle &= 0, \\
 d \langle b^2 G_{[]} (t) \rangle /dt &= -1 \langle b^3 G_{[]} (t) \rangle & \langle b^2 G_{[]} (0) \rangle &= \langle b^2 \rangle,
 \end{aligned} \tag{2.11}$$

We may close off this hierarchy at successively higher stages by taking the zeroth approximation that b and $G_{[]} (t)$ are statistically independent, and then admitting successively higher-order cumulants of the joint distribution (higher 'correlations' in the language of statistical field physics). Let us again assume (2.9). Then the appropriate successive closure approximations are

$$\begin{aligned}
 \langle bG_{[]} (t) \rangle &= \langle b \rangle G(t) = 0, \\
 \langle b^2 G_{[]} (t) \rangle &= \langle b^2 \rangle G(t), \\
 \langle b^3 G_{[]} (t) \rangle &= 3 \langle b^2 \rangle \langle bG_{[]} (t) \rangle, \\
 \langle b^4 G_{[]} (t) \rangle &= 6 \langle b^2 \rangle \langle b^2 G_{[]} (t) \rangle - 3 \langle b^2 \rangle^2 G(t),
 \end{aligned} \tag{2.12}$$

(Note that if $G_{[]} (t)$ were statistically independent of b then all these relations would be exact.) Using these relations in turn to close off (2.11) at successively higher stages, we obtain

$$\begin{aligned}
 G(t) &= 1, \\
 G(t) &= \cos(b_* t), \\
 G(t) &= \frac{2}{3} + \frac{1}{3} \cos(\sqrt{3} b_* t), \\
 G(t) &= \frac{1}{6} (3 + \sqrt{6}) \cos \left[(3 - \sqrt{6})^{1/2} b_* t \right] + \frac{1}{6} (3 - \sqrt{6}) \cos \left[(3 + \sqrt{6})^{1/2} b_* t \right]
 \end{aligned} \tag{2.13}$$

where $b_* = \langle b^2 \rangle^{1/2}$

Beyond the zeroth stage, which yields identical results in the two cases, the sequence (2.13) is distinctly superior to the approximations obtained by truncating (2.10). All members of the sequence satisfy (2.4) and (2.5). In common with the iteration scheme, the first n even derivatives of $G(t)$ at $t = 0$ are correct in the n th approximation. However, there still is no uniform validity in the sense $G(t) \rightarrow 0, t \rightarrow \infty$. None of the moments $\int_0^\infty t^n G(t) dt$ ($n = 1, 2, \dots$) exist for any approximation in the sequence, whereas they all do for the exact solution. Alternatively, we may note that $\tilde{G}(\omega)$, which is smooth in the exact solution, is a sum of δ functions in any of the cumulant-discard approximations. The convergence to the exact $G(t)$ is still very poor for $t > 2/b_*$.

It is clear that the random oscillator exhibits in acute form certain shortcomings of the iteration (perturbation) and cumulant-discard approaches to dynamical equations which are nonlinear in stochastic quantities. Both for this reason and because of its simplicity, we shall use the random oscillator to illustrate the alternative approach which is the subject of this paper. The sensitivity to inadequacies in the method of approximation arises because the solution to the 'unperturbed' equation, obtained by replacing the right side of (2.2) with zero, has a monochromatic spectrum. In this respect it resembles certain limiting cases of statistical field theory problems which are of current interest and to which our approach will be applicable. Examples are
 a quantum-mechanical particle in a random potential in the WKB limit, turbulence at infinite Reynolds number, and, in a less direct sense, a second-quantized many-boson system at very low temperature.

3. COLLECTIVE REPRESENTATION FOR A SET OF OSCILLATORS

We shall now describe a dynamical representation which is appropriate for formulating the stochastic models promised in Section 1. We shall introduce the representation formally and then give a physical interpretation and a comparison with more familiar concepts.

In Section 2 we treated an ensemble of realizations of a single oscillator. Now let us consider a collection of M oscillators ($M = 2S + 1$, $S =$ positive integer) whose frequencies are identically and independently distributed over an ensemble of realizations of the collection. We shall be interested in the limit $M \rightarrow \infty$, so that in reality we are introducing a two-dimensional distribution, so to speak. In place of the frequencies and amplitudes of the M individual oscillators, let us adopt the collective parameters and coordinates

$$\begin{aligned} b_a &= M^{-1/2} \sum_n \exp(i2\pi an/M) b[n], \\ q_a(t) &= M^{-1/2} \sum_n \exp(i2\pi an/M) q[n](t), \end{aligned} \quad (3.1)$$

$$(a = 0, \pm 1, \dots, \pm S) \quad (n = 1, 2, \dots, M),$$

where $b[n]$ and $q[n](t)$ are the frequency and amplitude of the n th oscillator.⁹

The identities

$$M^{-1} \sum_a \exp[i2\pi a(n-m)/M] = \delta_{n,m}, \quad M^{-1} \sum_n \exp[i2\pi n(a-\beta)/M] = \delta_{a,\beta} \quad (3.2)$$

yield

$$b_{[n]} = M^{-1/2} \sum_{\alpha} \exp(-i2\pi\alpha n/M) b_{\alpha},$$

$$q_{[n]}(t) = M^{-1/2} \sum_{\alpha} \exp(-i2\pi\alpha n/M) q_{\alpha}(t). \quad (3.3)$$

Let us adopt hereafter the cyclic convention $\alpha + M \equiv \alpha$, which clearly is consistent with (3.1) and (3.3).

From (3.1) and (3.2) we easily find

$$b_{-\alpha} = b_{\alpha}^*, \quad \sum_{\alpha} b_{\alpha} b_{-\alpha} = \sum_n b_{[n]}^2,$$

$$\sum_{\alpha} q_{\alpha}(t) q_{\alpha}^*(t') = \sum_n q_{[n]}(t) q_{[n]}^*(t'), \quad (3.4)$$

where we have used $b_{[n]} = b_{[n]}^*$. From the equations of motion

$$dq_{[n]}(t)/dt + i b_{[n]} q_{[n]}(t) = 0, \quad (3.5)$$

we find

$$dq_{\alpha}(t)/dt = -iM^{-1/2} \sum_{\beta} b_{\beta} q_{\alpha-\beta}(t), \quad (3.6)$$

where $\alpha - \beta$ is to be interpreted according to the cyclic convention. This shows that the new coordinates, in contrast to the old, are dynamically coupled. By (3.4), $\sum_{\alpha} q_{\alpha}(t) q_{\alpha}^*(t)$ is a constant of motion. Let $G_{[n,m]}(t)$ denote the solution of (3.5) with $q_{[r]}(0) = \delta_{r,m}$ and let $G_{\alpha,\gamma}(t)$ denote the solution of (3.6) with $q_{\mu}(0) = \delta_{\mu,\gamma}$. Since the individual oscillators are uncoupled, we have

$G_{[n,m]}(t) = \delta_{n,m} G_{[\bar{n},\bar{n}]}(t)$. Hence, by (3.1), (3.3), and the linearity of the equations of motion, the response matrices in the two representations are related by

$$G_{\alpha,\gamma}(t) = M^{-1} \sum_n \exp[i2\pi(\alpha-\gamma)n/M] G_{[n,n]}(t). \quad (3.7)$$

The statistical properties of the b_α are easily found from the assumption that the $b_{[n]}$ are identically and independently distributed. By (3.1) and (3.2), we have immediately

$$\langle b_\alpha b_\beta \dots b_\mu \rangle = 0 \quad (\alpha + \beta + \dots + \mu \neq 0). \quad (3.8)$$

For Gaussian $b_{[n]}$ all odd-order moments vanish, and we have

$$\langle b_{[n]} b_{[m]} \rangle = \delta_{n,m} \langle b^2 \rangle,$$

$$\langle b_{[n]} b_{[m]} b_{[r]} b_{[s]} \rangle = (\delta_{n,m} \delta_{r,s} + \delta_{n,r} \delta_{m,s} + \delta_{n,s} \delta_{m,r}) \langle b^2 \rangle^2, \quad (3.9)$$

.

where $\langle b^2 \rangle = \langle b_{[n]}^2 \rangle$, which is the same for all n . Then, by (3.1),

$$\langle b_\alpha \dots b_\mu \rangle = 0 \quad (\text{odd number of factors}),$$

$$\langle b_\alpha b_\beta \rangle = \delta_{\alpha+\beta} \langle b^2 \rangle \quad (\delta_{\alpha+\beta} = \delta_{0,\alpha+\beta}),$$

$$\langle b_\alpha b_\beta b_\gamma b_\epsilon \rangle = (\delta_{\alpha+\beta} \delta_{\gamma+\epsilon} + \delta_{\alpha+\gamma} \delta_{\beta+\epsilon} + \delta_{\alpha+\epsilon} \delta_{\beta+\gamma}) \langle b^2 \rangle^2, \quad (3.10)$$

.

For any univariate $b_{[n]}$ distribution we find from (3.7),

$$\langle G_{\alpha,\beta}(t) \rangle = \delta_{\alpha,\beta} G(t),$$

$$\langle b_{\beta} G_{\gamma,\alpha}(t) \rangle = \delta_{\alpha,\beta+\gamma} M^{-1/2} \langle b G_{[\gamma]}(t) \rangle,$$

$$\langle G_{\alpha,\gamma}(t) G_{\gamma,\alpha}^*(t) \rangle = \delta_{\alpha,\gamma} |G(t)|^2 + M^{-1} \left[\langle |G_{[\gamma]}(t)|^2 \rangle - |G(t)|^2 \right]. \quad (3.11)$$

Here $G_{[\gamma]}(t)$ denotes $G_{[n,n]}(t)$, which is the same for all n , and $G(t) = \langle G_{[\gamma]}(t) \rangle$, as in Section 2. In the limit $M \rightarrow \infty$, with which we are concerned, (3.11) gives

$$\langle G_{\alpha,\alpha}(t) \rangle = G(t), \quad \langle |G_{\alpha,\alpha}(t) - G(t)|^2 \rangle = O(M^{-1}),$$

$$\langle G_{\alpha,\gamma}(t) G_{\gamma,\alpha}^*(t) \rangle = O(M^{-1}), \quad \langle b_{\gamma} G_{\alpha-\gamma,\alpha}(t) \rangle = O(M^{-1/2}) \quad (\alpha \neq \gamma). \quad (3.12)$$

These relations show that the variance of $G_{\alpha,\alpha}(t) = M^{-1} \sum_n G_{[n,n]}(t)$ vanishes in the limit. That is, $G_{\alpha,\alpha}(t)$ is statistically sharp. They further imply that the effective dynamical coupling between any given pair of degrees of freedom q_{α} and q_{γ} is infinitely weak in the limit. Eqs. (3.12) were obtained without explicit reference to (3.6) but their dynamical implications may also be inferred from the latter. The direct dynamical coupling of q_{α} to any q_{γ} arises from only one of the M terms, each $O(M^{-1/2})$, on the right side of (3.6). Eqs. (3.12) show that the effective coupling is still $O(M^{-1/2})$ when the indirect interaction of q_{α} and q_{γ} through all the other degrees of freedom is included. The absence of fluctuations in $G_{\alpha,\alpha}(t)$ in the limit is consistent with the fact that this function is determined by the simultaneous interaction of q_{α} with an infinite number of other degrees of freedom; q_{α} exhibits negligible self-coupling, in contrast to $q_{[n]}$.

It is apparent, both from (3.1) and the convolution structure of (3.6), that the q_α have a close formal relation to Fourier coefficients. The physical significance of the new representation is best brought out, in fact, by a comparison with analysis into wavenumber or frequency components. Let $\psi(x,t)$ be a scalar field, associated with an extended (one-dimensional) dynamical system, which is described by an ensemble statistically invariant under translation (e.g., $\langle \psi(x,t)\psi(x',t) \rangle = \langle \psi(x+y,t)\psi(x'+y,t) \rangle$ for all y). The natural coordinates for describing the field then are wavenumber components, which change only by a phase factor under translation. Suppose, instead, the ensemble were invariant under time displacement. Then the natural coordinates would be frequency components.

Physical systems usually are neither statistically homogeneous nor stationary. However, if we form a collection of identically distributed individual systems, then obviously (and, it will appear at first sight, trivially) there is statistical invariance under permutation within the collection. The new representation is natural in the presence of this invariance in the same way that a wavenumber representation is natural when there is translational invariance.

To examine the analogy further, let us take

$$\psi_k(t) = L^{-1} \int_{-L/2}^{L/2} \psi(x,t) \exp(-ikx) dx \quad (k=2\pi\alpha/L, \alpha=0, \pm 1, \dots),$$

where we adopt the customary device of making the field cyclic with a period L which is as large as we wish compared to any relevant correlation length. Let

us divide L into very many segments, each still very large compared to any correlation length. Then each segment contains a sub-system which has only a negligible statistical dependence on its neighbors. Furthermore, it is plausible to suppose that (over times which are not too large) each sub-system has only a negligible dynamical interaction with its neighbors. Then we validly may regard the set of sub-systems as analogous to the collection of perfectly independent systems used above in defining the q_a . Considered in this way, the ψ_k and the q_a (for large M) play essentially similar roles. Both are linear combinations of the physical coordinates of a very large number of effectively independent systems.¹¹

4. FORMULATION OF MODEL PROBLEMS

Consider, instead of (3.6), the more general equations

$$dq_a(t)/dt = -iM^{-1/2} \sum_{\beta} \phi_{a,\beta,a-\beta} b_{\beta} q_{a-\beta}(t) , \quad (4.1)$$

where $\phi_{a,\beta,a-\beta}$ is independent of t and the same for every realization in the ensemble. We shall be interested in stochastic assignments of $\phi_{a,\beta,a-\beta}$ in the sense that this quantity will exhibit random changes in value as a and β are changed. By (3.3) and (3.2), (4.1) implies

$$dq_{[n]}(t)/dt = -i \sum_{r,s} A_{[n,r,s]} b_{[r]} q_{[s]}(t) , \quad (4.2)$$

where

$$A_{[n,r,s]} = M^{-2} \sum_{\beta,\gamma} \exp\left\{i2\pi[\beta(r-n) + \gamma(s-n)]/M\right\} \phi_{\beta+\gamma,\beta,\gamma} . \quad (4.3)$$

Thus, the individual oscillators in the collection now are dynamically coupled.

When $\phi_{\alpha,\beta,\alpha-\beta} = 1$ for all α and β , then $A_{[n,r,s]} = \delta_{r,n} \delta_{s,n}$ so that we recover the original collection of uncoupled oscillators. The quantities $q_{[n]}(t)q_{[n]}^*(t)$ are no longer constants of motion in the general case. However, we shall require

$$\phi_{\alpha,\beta,\alpha-\beta} = \phi_{\alpha-\beta,-\beta,\alpha}^* \quad (4.4)$$

Then, since $b_{-\beta} = b_{\beta}^*$, we find

$$d(\sum_n q_{[n]} q_{[n]}^*)/dt = d(\sum_{\alpha} q_{\alpha} q_{\alpha}^*)/dt = 0 \quad (4.5)$$

The response matrix corresponding to (4.1) satisfies

$$dG_{\alpha,\gamma}(t)/dt = -iM^{-1/2} \sum_{\beta} \phi_{\alpha,\beta,\alpha-\beta} b_{\beta} G_{\alpha-\beta,\gamma}(t), \quad G_{\alpha,\gamma}(0) = \delta_{\alpha,\gamma} \quad (4.6)$$

Suppose that we carry out a formal iteration expansion of (4.6). The coefficient of t^n in the resulting power series for $G_{\alpha,\gamma}(t)$ is a sum over products of n factors ϕ and n factors b . It is clear, from the initial condition and the way the indices combine, that in each product the sum of the indices of the b factors must be $\alpha - \gamma$. Hence, by (3.8),

$$\langle G_{\alpha,\gamma}(t) \rangle = 0 \quad (\alpha \neq \gamma) \quad (4.7)$$

By (3.1), (3.3), and (4.7), we have

$$\langle G_{[n,m]}(t) \rangle = M^{-1} \sum_{\alpha} \exp[-i2\pi(n-m)\alpha/M] \langle G_{\alpha,\alpha}(t) \rangle \quad (4.8)$$

Therefore, if $\langle G_{a,a}(t) \rangle$ is independent of a , we have, by (3.2),

$$G_{[n,m]}(t) = \delta_{n,m} G(t), \quad G(t) = \langle G_{a,a}(t) \rangle, \quad (4.9)$$

where $G(t)$ has now the same meaning as in Section 3. We shall be concerned only with $\emptyset$ assignments which yield (4.9), and presently we shall exhibit certain of their properties.

We wish now to develop an expression for $d \langle G_{a,a}(t) \rangle / dt$ by expanding the right side of (4.6), with $\gamma = a$. Writing

$$H_{a-\beta,\beta,a}(t) = -iM^{-1/2} \emptyset_{a,\beta,a-\beta} b_{\beta} G_{a-\beta,a}(t), \quad (4.10)$$

we may develop $\langle H_{a-\beta,\beta,a}(t) \rangle$ in powers of t by using the iteration expansion for $G_{a-\beta,a}(t)$. The coefficient of t^{r-1} in the resulting series is a sum, over all the indices of $\wedge$ b factors except b_{β} , of products of r factors b and r factors $\emptyset$ [including the factors b_{β} and $\emptyset_{a,\beta,a-\beta}$ which appear explicitly in (4.10)]. Let us take Gaussian $b_{\text{and (3.10)}}[n]$. (We shall return to the general case in Section 9.) Then, by (3.8), only the odd powers of t survive and in the coefficients of these the indices of the b factors must be equal and opposite in pairs. In this way we find

$$\langle H_{a-\beta,\beta,a}(t) \rangle = M^{-1} \sum_{n=1}^{\infty} \sum_p (-1)^n c_{2n;p}(a,\beta,a-\beta) \langle b^2 \rangle^n t^{2n-1} / (2n-1)!, \quad (4.11)$$

where $M^{-1} c_{2n;p}(a,\beta,a-\beta)$ is a sum of products of $2n$ factors $\emptyset$ and the index p ($p=1,2,\dots,2n!/2^n n!$) labels (in an arbitrary order) the contributions which arise from all the possible pairings of the b factors.

The $C_{2n;p}(\alpha, \beta, \alpha-\beta)$ through $n = 2$ are

$$C_{2;1}(\) = \phi_{\alpha, \beta, \alpha-\beta} \phi_{\alpha-\beta, -\beta, \alpha},$$

$$C_{4;1}(\) = M^{-1} \sum_{\gamma} \phi_{\alpha, \beta, \alpha-\beta} \phi_{\alpha-\beta, -\beta, \alpha} \phi_{\alpha, \gamma, \alpha-\gamma} \phi_{\alpha-\gamma, -\gamma, \alpha},$$

$$C_{4;2}(\) = M^{-1} \sum_{\gamma} \phi_{\alpha, \beta, \alpha-\beta} \phi_{\alpha-\beta, \gamma, \alpha-\beta-\gamma} \phi_{\alpha-\beta-\gamma, -\gamma, \alpha-\beta} \phi_{\alpha-\beta, -\beta, \alpha},$$

$$C_{4;3}(\) = M^{-1} \sum_{\gamma} \phi_{\alpha, \beta, \alpha-\beta} \phi_{\alpha-\beta, \gamma, \alpha-\beta-\gamma} \phi_{\alpha-\beta-\gamma, -\beta, \alpha-\gamma} \phi_{\alpha-\gamma, -\gamma, \alpha} \quad (4.12)$$

The ϕ factors in (4.12) other than $\phi_{\alpha, \beta, \alpha-\beta}$ are written from the right in the order they arise in the iteration process. (The values of p are assigned arbitrarily.)

Each $C_{2n;p}(\alpha, \beta, \alpha-\beta)$ is closed in the sense that the initial index on any factor equals the final index on the factor to its immediate left [when ordered as in (4.12)] while the middle indices are equal and opposite in pairs. This permits a systematic diagrammatic representation of the $C_{2n;p}(\alpha, \beta, \alpha-\beta)$ and therefore of (4.11). With each $\phi_{\mu, \lambda, \sigma}$ or $\phi_{\mu, \lambda, \sigma}^*$ let us associate a vertex as shown in Figure 1. The $C_{2n;p}(\alpha, \beta, \alpha-\beta)$ then may be obtained by the following rules:

Connect $2n$ points by $2n$ solid line segments to form a single closed loop; then connect all the points in pairs by n dashed line segments to form a closed diagram of $2n$ vertices. Equip all the solid line segments with arrows pointing in the same sense. Choose one vertex, equip its dashed line segment with an ingoing arrow, and label its three line segments to correspond to $\phi_{\alpha, \beta, \alpha-\beta}$ in the sense of Fig. 1a. Call this the fixed vertex and identify it by circling. Label the remaining dashed line segments $\gamma, \epsilon, \dots$ in any order and equip them with arrows (whose

direction does not matter¹²). Complete the labelling of the solid line segments so that the sum of the indices labelling the ingoing lines equals the sum of those labelling the outgoing lines at every vertex. Now write the product of the ϕ factors associated with all the vertices according to Fig. 1a, or according to Fig. 1b and (4.4). For each n there are $2n!/2^{n!}$ distinct diagrams of this type, corresponding to the $2n!/2^{n!}$ ways of connecting the vertices by dashed lines after one is chosen as the fixed vertex. Each diagram corresponds to one $C_{2n;p}(\alpha, \beta, \alpha - \beta)$ (according to an arbitrary rule for assigning the values of p) and the latter is equal to M^{1-n} times the sum over $\gamma, \epsilon, \dots$ of the associated ϕ product.

The diagrams associated with (4.12) are shown in Fig. 2.

It is clear from (4.11) that $\langle G_{\alpha, \alpha}(t) \rangle$ will be independent of α , and therefore (4.9) will hold, if

$$M^{-1} \sum_{\beta} C_{2n;p}(\alpha, \beta, \alpha - \beta) = C_{2n;p}, \quad (4.13)$$

where $C_{2n;p}$ is independent of α . In this case,

$$G(t) = 1 + \sum_{n=1}^{\infty} \sum_p (-1)^n C_{2n;p} \langle b^2 \rangle_t^{2n/2n!}. \quad (4.14)$$

We shall be concerned hereafter only with ϕ assignments such that (4.13) is satisfied when $M \rightarrow \infty$. The $C_{2n;p}$ may be interpreted as moments of the distribution of the quantity $\phi_{\mu, \lambda, \mu - \lambda}$ over the set of index values μ and λ .

Let us associate with $C_{2n;p}$ the diagram for $C_{2n;p}(\alpha, \beta, \alpha - \beta)$, but with index labels and dashed line arrows (which are now superfluous) omitted. By (4.13) we have $C_{2n;p} = M^{-2} \sum_{\alpha, \beta} C_{2n;p}(\alpha, \beta, \alpha - \beta)$. Recalling the cyclic convention $\mu \equiv \mu + M$ (any μ), we see that the summation in this expression is equivalent to one over all M values of all the indices labelling lines in the diagram, subject only to the sum condition at each vertex. Consequently, the expression

is independent of which is the fixed vertex; its value depends only on the order and topology of the diagram.

It is important to point out that (3.12) is valid for general ϕ 's satisfying (4.13). In particular, $G_{\alpha,\alpha}(t)$ is statistically sharp ($M \rightarrow \infty$). In the original case (all ϕ 's unity) this was so because q_α interacted simultaneously with all the other degrees of freedom, and negligibly with itself. These properties clearly characterize the general case also, provided the ϕ 's are bounded as $M \rightarrow \infty$. The validity of (3.12) in the general case is easily demonstrated¹³ for any power of t in the iteration expansions of the left sides of the equations, if one uses (3.10) and the fact that the ϕ 's are the same for all realizations. We shall use the abbreviations C and $C()$ to denote $C_{2n;p}$ and $C_{2n;p}(\alpha,\beta,\alpha-\beta)$, respectively, when it is not desired to specify particular subscripts and arguments.

Let us define a reducible C as one which may be factored into two or more C 's of lower order, and an irreducible C as one which may not. Let us define a reducible $C()$ as one which may be factored into the product of a lower-order $C()$ with one or more C 's, and an irreducible $C()$ as one which may not. It follows that each reducible C is a product of irreducible C 's and each reducible $C()$ is a product of an irreducible $C()$ with irreducible C 's. It is easy to see from our rules that reducible $C()$'s and C 's (and only they) are associated with diagrams in which there is a part, or parts, connected to the rest of the diagram by only solid lines. Thus, $C_{2;1}$ and $C_{4;3}$ are irreducible, but $C_{4;1}$ and $C_{4;2}$ are reducible. Using (4.13) we find $C_{4;1} = C_{4;2} = (C_{2;1})^2$.

Let us write each $C()$ which appears in (4.11) as the product of an irreducible $C()$ and irreducible C 's, and then collect all the terms proportional

to each irreducible $C()$. We obtain a result of the form

$$\langle H_{\alpha-\beta,\beta,\alpha}(t) \rangle = M^{-1} \sum_n \sum_p^{\text{irr}} (-1)^n C_{2n;p}(\alpha,\beta,\alpha-\beta) \zeta_{2n;p}(t), \quad (4.15)$$

where $\sum^{\text{irr}}$ denotes the sum over irreducible diagrams only. The $\zeta_{2n;p}(t)$ depend on the values of the irreducible C 's but are independent of α and β . Each $\zeta_{2n;p}(t)$ contains all (odd) powers of t which are $\geq 2n-1$, since each $C_{2n;p}(\alpha,\beta,\alpha-\beta)$ appears in reducible $C()$'s of all orders $\geq 2n$. The $\zeta_{2n;p}(t)$ turn out to have simple expressions in terms of $\langle b^2 \rangle$ and $G(t)$ which may be found by comparing the explicit power series for $\zeta_{2n;p}(t)$ and $G(t)$. However, the same result may be obtained more transparently by a variational procedure which provides certain dynamical insights.

There are $M(M-1)$ ϕ 's, and only M sums $\sum_{\beta} C_{2n;p}(\alpha,\beta,\alpha-\beta)$ for given n and p . In the limit $M \rightarrow \infty$ it will be possible, therefore, to make wide classes of variations $\Delta \phi$ such that (4.4) and (4.13) continue to hold and such that $\Delta C_{2n;p} = 0$ for all finite n . Under these constraints, $\Delta \zeta_{2n;p}(t) = 0$. Consequently, we have

$$\Delta \langle H_{\alpha-\beta,\beta,\alpha}(t) \rangle = M^{-1} \sum_n \sum_p^{\text{irr}} (-1)^n \zeta_{2n;p}(t) \Delta C_{2n;p}(\alpha,\beta,\alpha-\beta). \quad (4.16)$$

Now consider a (finite) variation $\Delta \phi_{\alpha-\beta,-\beta,\alpha} = \Delta \phi_{\alpha,\beta,\alpha-\beta}^*$ for a particular α and β , with all the other ϕ 's fixed. We may vary the real and imaginary parts of $\phi_{\alpha-\beta,-\beta,\alpha}$ independently. Identical results are obtained by supposing $\phi_{\alpha-\beta,-\beta,\alpha}$ to vary while $\phi_{\alpha-\beta,-\beta,\alpha}^* = \phi_{\alpha,\beta,\alpha-\beta}$ is held fixed, and we shall adopt the latter procedure. Then, by (4.12),

$$\Delta C_{2;1}(\alpha,\beta,\alpha-\beta) = \phi_{\alpha,\beta,\alpha-\beta} \Delta \phi_{\alpha-\beta,-\beta,\alpha}, \quad (4.17)$$

while from (4.13) we see that $\Delta C_{2n;p} = O(M^{-1})$ for all n and p . Thus the constraints stated above are satisfied for $M \rightarrow \infty$. By (4.6), $G_{\alpha-\beta, \alpha}(t)$ satisfies

$$dG_{\alpha-\beta, \alpha}(t)/dt + iM^{-1/2} \sum_{\sigma} \phi_{\alpha-\beta, -\sigma, \alpha-\beta+\sigma} b_{-\sigma} G_{\alpha-\beta+\sigma, \alpha}(t) = 0. \quad (4.18)$$

The effect of the variation $\Delta \phi_{\alpha-\beta, -\beta, \alpha}$ is to produce on the right side of (4.18) the additional term

$$-iM^{-1/2} \Delta \phi_{\alpha-\beta, -\beta, \alpha} b_{-\beta} G_{\alpha, \alpha}(t)$$

which, we note, is $O(M^{-1/2})$. Now we recall that $G_{\alpha-\beta, \alpha}(t)$ is simply the amplitude $q_{\alpha-\beta}(t)$ under a particular initial condition at $t = 0$. Therefore, to order $M^{-1/2}$, we have

$$\Delta G_{\alpha-\beta, \alpha}(t) = \int_0^t G_{\alpha-\beta, \alpha-\beta}(t-s) \left[-iM^{-1/2} \Delta \phi_{\alpha-\beta, -\beta, \alpha} b_{-\beta} G_{\alpha, \alpha}(s) \right] ds, \quad (4.19)$$

since $G_{\alpha-\beta, \alpha-\beta}$ is the diagonal response function for $q_{\alpha-\beta}$ and the perturbation does not affect the initial condition. [Note that $\Delta G_{\alpha, \alpha}(t)$ and $\Delta G_{\alpha-\beta, \alpha-\beta}(t)$ are $O(M^{-1})$ under our constraints.] Referring to (4.10), we obtain

$\Delta \langle H_{\alpha-\beta, \beta, \alpha}(t) \rangle$ immediately. It is clear that our variation gives $\Delta C_{2n;p}(\alpha, \beta, \alpha-\beta) = O(M^{-1})$ for all irreducible diagrams with $n > 1$. Then by (4.16) and (4.17) we find

$$\zeta_{2;1}(t) = \int_0^t \langle b_{\beta} b_{-\beta} G_{\alpha-\beta, \alpha-\beta}(t-s) G_{\alpha, \alpha}(s) \rangle ds \quad (4.20)$$

in the limit $M \rightarrow \infty$. As we have noted previously, $G_{\alpha, \alpha}(t)$ and $G_{\alpha-\beta, \alpha-\beta}(t)$

are statistically sharp in the limit. Therefore, by (4.9) and (3.10),

$$\zeta_{2;1}(t) = \langle b^2 \rangle \int_0^t G(t-s)G(s)ds. \quad (4.21)$$

The higher $\zeta_{2n;p}$ may be found by similar analysis based on more general variations. The result is

$$\zeta_{2n;p} = \langle b^2 \rangle^n G(*G)^{2n-1}, \quad (4.22)$$

where $G(*G)^{2n-1}$ is a repeated convolution; e.g. (for argument t),

$$G(*G)^3 = \int_0^t ds \int_0^s ds' \int_0^{s'} ds'' G(t-s)G(s-s')G(s'-s'')G(s'').$$

Collecting the appropriate relations, we have the final result

$$dG/dt = \sum_{n=1}^{\infty} \sum_p^{irr} (-1)^n C_{2n;p} \langle b^2 \rangle^n G(*G)^{2n-1}, \quad G(0) = 1. \quad (4.23)$$

The value of this infinite-series, integro-differential equation for $G(t)$ is that only the irreducible $C_{2n;p}$ appear explicitly.

5. THE RANDOM COUPLING MODEL

We shall now consider a particular stochastic assignment of the ϕ 's. Let

$$\phi_{\alpha,\beta,\alpha-\beta} = \exp(i\theta_{\alpha,\beta,\alpha-\beta}), \quad (5.1)$$

where $\theta_{\alpha,\beta,\alpha-\beta}$ is real and satisfies

$$\theta_{\alpha-\beta,-\beta,\alpha} = -\theta_{\alpha,\beta,\alpha-\beta}. \quad (5.2)$$

For each choice of α and β , let $\theta_{\alpha,\beta,\alpha-\beta}$ take a value at random in the interval 0 to 2π , subject only to (5.2). The value must be the same, of course, for every realization in the ensemble. Now let $M \rightarrow \infty$. Clearly this assignment satisfies (4.4). In addition, it yields $|\phi_{\alpha,\beta,\alpha-\beta}| = 1$, and therefore retains unaltered the strengths of the individual dynamical couplings of pairs $q_\alpha, q_{\alpha-\beta}$ which characterize (3.6). Now, however, the phases of the couplings are completely unrelated for different pairs. We shall call the present choice the random coupling model.¹⁴

Referring to (4.12) and (4.13), we find $C_{2;1} = 1$, as in the true problem (all ϕ 's unity). Consider $C_{4;3}$, however. Each product in the sum has modulus 1, but the phase of the product changes at random with β and γ . Consequently, $C_{4;3} = 0(M^{-1})$. In a similar fashion we see that the only $C_{2n;p}$ which survive in the limit are those in which the product of ϕ 's consists entirely of conjugate pairs and which therefore are reducible to powers of $C_{2;1}$. Consequently, for $M \rightarrow \infty$,

$$C_{2;1} = 1,$$

$$C_{2n;p} = 0 \quad (n > 1) \quad (\text{all irreducible diagrams}). \quad (5.3)$$

It follows that (4.23) reduces to the closed form

$$dG/dt + \langle b^2 \rangle G * G = 0, \quad G(0) = 1. \quad (5.4)$$

Eq. (5.4) is readily solved by Laplace transformation. We find

$$G(t) = J_1(2b_*t)/b_*t, \quad (5.5)$$

$$\tilde{G}(\omega) = \begin{cases} \pi^{-1} [1 - (\omega/2b_*)^2]^{1/2} & (|\omega| \leq 2b_*) \\ 0 & (|\omega| > 2b_*) \end{cases}, \quad (5.6)$$

where $b_* = \langle b^2 \rangle^{1/2}$ as before. Considered as an approximation to (2.8) and (2.7), the present results display a type of uniform validity which is absent in any finite stage of the iteration or cumulant-discard schemes discussed in Section 2. The spectral density (5.6) is continuous, and (2.4) and (2.5) are satisfied. All the derivatives of $\tilde{G}(\omega)$ exist at $\omega = 0$ so that all the moments $\int_0^\infty t^n G(t) dt$ exist. The results (2.8) and (5.5) are compared in Fig. 3.

It is important that certain of these properties could have been predicted from the sole fact that (5.4) is an exact equation ($M \rightarrow \infty$) for a realizable $\emptyset$ assignment satisfying (4.4) and, hence, for a conservative dynamical problem. We recall that $G_{\alpha, \alpha}(t) = q_\alpha(t)$ when $q_Y(0) = \delta_{\alpha, Y}$. But then $\sum_Y q_Y(0) q_Y^*(0) = 1$, and, since $\sum_Y q_Y(t) q_Y^*(t)$ is a constant of motion, we have $|G_{\alpha, \alpha}(t)| \leq 1$, whence (2.5) readily follows. To establish (2.4), let us make, for each realization, a similarity transformation

$$q_\alpha(t) = \sum_Y B_{\alpha Y} q_Y'(t), \quad q_Y'(t) = \sum_\alpha B_{Y\alpha}^{-1} q_\alpha(t), \quad B_{Y\alpha}^{-1} = B_{\alpha Y}^*$$

($B_{\alpha Y}$ independent of t) such that (4.1) is brought to the diagonal form

$$dq_Y'(t)/dt + i\omega_Y q_Y'(t) = 0. \quad (5.7)$$

Since $\sum_Y q_Y' q_Y'^* = \sum_\alpha q_\alpha q_\alpha^*$ is a constant of motion, the ω_Y are real. Now if $G_{Y, \epsilon}'(t)$ is the response matrix of the new variables, $G_{Y, \epsilon}'(t) = \delta_{Y, \epsilon} \exp(-i\omega_Y t)$, and therefore

$$G_{\alpha, \alpha}(t) = \sum_{Y, \epsilon} B_{\alpha, Y} G_{Y, \epsilon}'(t) B_{\epsilon \alpha}^{-1} = \sum_Y B_{\alpha Y} B_{\alpha Y}^* \exp(-i\omega_Y t).$$

Hence $\tilde{G}_{\alpha, \alpha}(\omega)$ is real and non-negative in each realization, which implies (2.4). Finally, we note that the model problem resulting from a general, realizable $\emptyset$

assignment involves the interaction of an infinite number of degrees of freedom q_α when $M \rightarrow \infty$. From this, and the fact that the b_α are continuously distributed over the ensemble, we may anticipate that $\tilde{G}(\omega)$ exhibits a continuous or band structure and that $G(t) \rightarrow 0$, $t \rightarrow \infty$.

It is possible to understand qualitatively why the complex detailed dynamics of the random coupling model lead to a simplification of the statistical dynamics and to closure of (4.23). The function $\langle G_{\alpha,\alpha}(t) \rangle$ describes the decay of q_α due to transfer of an initial excitation $q_\alpha(0) = 1$ to the rest of the degrees of freedom. In general, the decay requires that certain phase relations be set up between q_α and the other amplitudes. A phase relation between q_α and $q_{\alpha-\beta}$ can arise either from direct dynamical coupling (involving the coupling coefficients $-iM^{-1/2} \phi_{\alpha,\beta,\alpha-\beta} b_\beta$ and $-iM^{-1/2} \phi_{\alpha-\beta,-\beta,\alpha} b_{-\beta}$) or from indirect coupling through chains of other modes q_γ . In fact, each term in the irreducible diagram expansion (4.15) may be regarded as describing the transfer of excitation from q_α to $q_{\alpha-\beta}$ along the chain of intermediate modes represented by the directed solid line segments in the associated diagram. The closing of the solid line on itself then represents the reaction on mode α and the consequent diminution of q_α . The factors G , whose repeated convolution yields the $\zeta_{2n,p}$, incorporate the effect of the dynamical interaction as a whole on the transfer process. This effect is to relax the phase relations set up along the chain.

In the random coupling model, only the direct interaction, associated with $C_{2;1}$, is effective in the transfer of excitation. The contributions associated with the indirect paths of interaction cancel, when summed over all possible intermediate modes, because of the random phases of the ϕ 's. The coupling of q_α and $q_{\alpha-\beta}$ to all the rest of the modes therefore affects $\langle H_{1-\beta,\beta,\alpha}(t) \rangle$ only by relaxing the phase relations induced by the direct interaction of these two modes.

All $C_{2n;p}$ which are expressible as powers of $C_{2;1}$ have the value unity in the random coupling model, and all other $C_{2n;p}$ vanish. Thus we see from (4.14) that the power series for the model $G(t)$ consists of a particular sub-set of terms of all orders from the corresponding series for the true problem (all $C_{2n;p}$ unity). The terms retained are all those whose associated diagrams can be reduced to that for $C_{2;1}$ (Fig. 2) by iterating any number of times, on any solid lines, the contraction operation shown in Fig. 4a. Examples of included diagrams are shown in Fig. 4b. It follows readily from (5.5) that the number of diagrams of this type with $2n$ vertices is

$$(-1)^n \left[\frac{d^{2n}}{(dt)^{2n}} \left(\frac{j_1(2t)}{t} \right) \right]_{t=0}.$$

6. INADMISSIBLE HIGHER APPROXIMATIONS

For the true problem (all $C_{2n;p} = 1$), (4.23) reads

$$dG/dt = \sum_{n=1}^{\infty} (-1)^n S_{2n} \langle b^2 \rangle^n G(*G)^{2n-1}, \quad G(0) = 1, \quad (6.1)$$

where S_{2n} is the number of irreducible diagrams of order $2n$. The first few S_{2n} are $S_2 = 1$, $S_4 = 1$, $S_6 = 4$, $S_8 = 27$.¹⁵ The relative success of (5.4) as an approximation to (6.1) suggests that we seek higher approximations satisfying equations of the form

$$dG/dt = \sum_{n=1}^R (-1)^n S_{2n} \langle b^2 \rangle^n G(*G)^{2n-1}, \quad G(0) = 1 \quad (R > 1), \quad (6.2)$$

which we obtain by giving all irreducible $C_{2n;p}$ the value one, $n \leq R$, and the

value zero, $n > R$. One property of (6.2) can be predicted immediately. We recall that (4.23) represents simply a consolidation of (4.14). The present sums $\sum_p C_{2n,p}$ (reducible and irreducible diagrams included) clearly do not exceed the corresponding sums in the true problem. Since (2.10), which constitutes (4.14) for that problem, is absolutely convergent for all t , it follows that the power series expansions of the solutions of (6.2) are absolutely convergent for all t .

Nevertheless, these solutions are not valid higher approximations to (2.8). The reason is that the functions to which their expansions converge become infinite as $t \rightarrow \infty$. None of them, therefore constitutes a uniform approximation, and for none of them does $\tilde{G}(\omega)$ exist. We shall illustrate this for $R = 2$. If $\hat{G}(p)$ denotes the Laplace transform of $G(t)$, then (6.2) for this case is equivalent to

$$p\hat{G}(p) = 1 - \langle b^2 \rangle [\hat{G}(p)]^2 + \langle b^2 \rangle^2 [\hat{G}(p)]^4. \quad (6.3)$$

Let us assume tentatively that $\hat{G}(0) = \int_0^\infty G(t)dt$ is finite. Then

$$[\hat{G}(0)]^2 = [1 \pm \sqrt{-3}]/2 \langle b^2 \rangle, \quad (6.4)$$

which is inconsistent with the reality of $G(t)$. Therefore, $\int_0^\infty G(t)dt$ cannot be finite. Further analysis readily shows that, for real ω , (6.3) is inconsistent with $\text{Re} \{ \hat{G}(-i\omega) \} = O(\omega^{-r})$, $\omega \rightarrow 0$, if r is any finite power. It follows that $G(t)$ grows faster than any power of t as $t \rightarrow \infty$.

The numerical solutions of (6.2) for several values of R are compared with (2.8) in Fig. 5. As R increases, it will be noted that the approximations increase in accuracy for small t but diverge faster at large t . In this respect, our present results resemble very closely those of truncating (2.10) after a

finite number of terms. They do not appear to represent a significant improvement over the latter.

The failure of the present approximations has an immediate interpretation. Divergence of $G(t)$ as $t \rightarrow \infty$ is inconsistent with (2.5). It follows that the values of the irreducible C 's implied by (6.2) are not realizable by any assignment of the ϕ 's consistent with (4.4). Thus these approximations do not correspond to any dynamic model in our sense.¹⁶

If we regard a stochastic assignment of the $\phi_{\alpha,\beta,\alpha-\beta}$ for $M \rightarrow \infty$ as a distribution over the set of index values α and β , then the $C_{2n;p}$ are moments of this distribution and there are an infinite set of realizability inequalities which they must satisfy. The values $C_{2;1} = 1$, $C_{2n;p} = 0$ (all higher irreducible diagrams) for the random coupling model correspond to complete statistical independence of the $\phi_{\alpha,\beta,\alpha-\beta} = \phi_{\alpha-\beta,-\beta,\alpha}^*$ in this sense. Nonvanishing values for the higher irreducible C 's imply statistical correlation among the ϕ 's.

The nature of realizability inequalities for simpler statistical problems suggests that when $C_{2;1} = 1$ a wide choice of realizable nonzero values can be given to the higher irreducible $C_{2n;p}$, provided these values are small enough. Let us consider the assignment

$$C_{2;1} = 1, \quad C_{4;3} = a, \quad C_{2n;p} = 0 \quad (\text{all higher irreducible diagrams}), \quad (6.5)$$

where a is a real constant. For $a = 1$ this gives (6.2), $R = 2$. Instead of (6.3), we now find, in the general case,

$$p\hat{G}(p) = 1 - \langle b^2 \rangle [\hat{G}(p)]^2 + a \langle b^2 \rangle^2 [\hat{G}(p)]^4, \quad (6.6)$$

whence

$$[\hat{G}(0)]^2 = [1 - (1 - 4a)^{1/2}] / 2a \langle b^2 \rangle \quad (6.7)$$

Eq. (6.6) yields real, non-negative $\tilde{G}(\omega)$ for all ω if

$$-1/12 \leq a \leq 1/4, \quad (6.8)$$

which suggests that (6.8) may represent the range of realizability of (6.5). In Fig. 6, the solution $\tilde{G}(\omega) = \pi^{-1} \operatorname{Re} \{ \hat{G}(-i\omega) \}$ on the relevant branch of (6.6) is compared, for several values of a , with (2.7) and (5.6). The form of $\tilde{G}(\omega)$ changes continuously with a up to the limit $a = 1/4$, where the slope at $\omega = 0$ changes abruptly from 0 to ∞ . It is apparent that none of the present approximations represents a substantial improvement over the random coupling solution ($a = 0$). For $a < 0$, the form of $\tilde{G}(\omega)$ changes continuously with a down to the limit $a = -1/12$; then, a singularity appears at the cut-off point of the spectrum. We conclude tentatively, lacking contrary evidence, that (6.8) does represent the range of realizability of (6.5).

The results of the present Section suggest that great caution be exercised in carrying out partial summations of diagrams in the power series expansion for $G(t)$. It is by no means true that the more terms summed, the better the approximation. Our inadmissible approximation (6.2) ($R = 2$) is equivalent to the retention, in (4.14) for the true problem, of all terms whose diagrams can be reduced to the diagram for $C_{2;1}$ by iterated application, on any solid lines, of the contraction operations shown in Fig. 7a. Examples are shown in Fig. 7b. Thus, the terms retained are selected according to well-defined and plausible topological properties of the diagrams. Moreover, as we have noted, in the t domain they constitute an absolutely convergent sub-series of an absolutely convergent series.

It will have been recognized by this point that the diagram summations we have employed are intimately related to summations of perturbation series terms in quantum field theory and quantum statistical mechanics. Our present results suggest that caution be exercised in these problems also. There too, it is possible that plausible-appearing and summable classes of diagrams are better omitted than included. We hope to return to these questions in a later paper.

7. SECOND STOCHASTIC MODEL

The results of the last Section emphasize the desirability of seeking higher approximations to (6.1) which correspond to realizable values of the C 's. We shall now describe a second stochastic model for which $\tilde{G}(\omega)$ satisfies (2.4) and is substantially closer to (2.7) than is the random coupling result. Consider the contraction operation shown in Fig. 8a. Each application to a diagram reduces the number of vertices by two. Let us take $C_{2;1} = 1$ and assign the value a^{n-1} to all irreducible $C_{2n;p}$ whose diagrams can be transformed into that for $C_{2;1}$ by $n-1$ applications, anywhere, of this operation. These diagrams represent an infinite subset of the terms in (4.23). (Examples are shown in Fig. 8b.) Let us assign the value zero to all other irreducible $C_{2n;p}$. (Examples are shown in Fig. 8c.) Now let us take $a = 1$. Clearly this implies the value 1, as in the true problem, for all reducible and irreducible $C_{2n;p}$ whose diagrams can be transformed into that for $C_{2;1}$ by repeated application of the line operation of Fig. 4a and the vertex operation of Fig. 8a. All other $C_{2n;p}$ have the value zero.

We have not found an explicit construction for this model of the type provided by (5.1), et seq., for the random coupling model. Consequently we have no proof of realizability. However, examination of the dependence of $\tilde{G}(\omega)$ on

a, as in Section 6, suggests that the model is realizable, as we shall see.

Since the present model retains an infinite subset of terms in (4.23), it does not directly yield a closed equation for $G(t)$. However, we can obtain a closed system [eqs. (7.6) and (7.17)] by considering simultaneously the first two equations of an hierarchy analogous to (2.11). Let

$$H(t) \equiv \sum_{\beta} \langle H_{\alpha-\beta, \beta, \alpha}(t) \rangle = dG(t)/dt .$$

From (4.6) we find

$$dH(t)/dt = - \sum_{\beta} \phi_{\alpha, \beta, \alpha-\beta} \phi_{\alpha-\beta, -\beta, \alpha} \langle b_{\beta} b_{-\beta} G_{\alpha, \alpha}(t) \rangle + J(t), \quad H(0) = 0, \quad (7.1)$$

where

$$J(t) = - M^{-1} \sum_{\beta, \gamma}^{(\beta \neq \gamma)} \phi_{\alpha, \beta, \alpha-\beta} \phi_{\alpha-\beta, \gamma, \alpha-\beta-\gamma} \langle b_{\beta} b_{\gamma} G_{\alpha-\beta-\gamma, \alpha}(t) \rangle . \quad (7.2)$$

Since $G_{\alpha, \alpha}(t)$ is statistically sharp ($M \rightarrow \infty$), it follows from previous relations that the first term on the right side of (7.1) may be rewritten $-C_{2;1} \langle b^2 \rangle G(t)$. Hence, when $C_{2;1} = 1$, we have the equations

$$\begin{aligned} dG(t)/dt &= H(t), & G(0) &= 1, \\ dH(t)/dt &= - \langle b^2 \rangle G(t) + J(t), & H(0) &= 0. \end{aligned} \quad (7.3)$$

By (4.23) we have

$$H = \sum_{n=1}^{\infty} \sum_p^{irr} (-1)^n C_{2n;p} \langle b^2 \rangle^n G(\ast G)^{2n-1}. \quad (7.4)$$

An analogous expansion for $J(t)$ may be found by analysis very similar to that which leads to (4.23). The result is

$$J = \sum_{n=2}^{\infty} \sum_p^J (-1)^{n_C} c_{2n;p} \langle b^2 \rangle^{n_C} (*G)^{2n-2}, \quad (7.5)$$

where $\sum^J$ is defined as follows: Construct a diagram part as shown in Fig. 9. Call it the fixed part. [The two vertices correspond to the two ϕ factors which appear explicitly in (7.2)] The summation $\sum^J$ then is over every closed diagram, constructed by combining two or more vertices with the fixed part, that contains no part, wholly external to the fixed part, which is connected to the rest of the diagram by only solid lines. The three simplest diagrams included are shown in Fig. 10. It will be noted that $\sum^J$ contains both reducible and irreducible diagrams in the sense of Section 4. The reducible diagrams all resemble Fig. 10a in that the associated C is the product of just two irreducible C 's.

Let us denote by Fig. 11a the totality of possible diagram parts, with the two solid and single dashed external lines shown, that can be transformed into a single vertex by repeated contractions as shown in Fig. 8a. Let us call this structure a consolidated vertex part. Then we may represent the entire class of irreducible diagrams which contribute to $H(t)$, in the present model, by the single consolidated diagram shown in Fig. 11b.¹⁷ In a similar fashion, we may represent by consolidated diagrams all the contributions to $J(t)$ in the present model. Clearly, two consolidated diagrams included are those shown in Figs. 12a and 12b. If we independently replace the several consolidated vertex parts in these diagrams by all possible diagram parts which they represent, then we obtain all the individual contributing diagrams which are contractible into Figs. 10a

or 10b, respectively. The required contractions do not involve the fixed part. It can be seen that all other diagrams contributing in the present model are represented by the infinite class of consolidated diagrams shown in Fig. 12c. These diagrams all may be transformed into that for $C_{2;1}$ by sequences of contractions which now involve the fixed part.

As the representation by consolidated diagrams suggests, the present model leads to a closed expression for $J(t)$ in terms of $H(t)$ and $G(t)$. It is convenient at this point to work with the Laplace transform representation. If $\hat{G}(p)$, $\hat{H}(p)$, and $\hat{J}(p)$ denote the respective transforms of $G(t)$, $H(t)$, and $J(t)$, then the transforms of (7.3)-(7.5) are¹⁸

$$\begin{aligned} p\hat{G}(p) &= 1 + \hat{H}(p) , \\ p\hat{H}(p) &= - \langle b^2 \rangle \hat{G}(p) + \hat{J}(p) , \end{aligned} \quad (7.6)$$

$$\hat{H}(p) = \sum_{n=1}^{\infty} \sum_p^{irr} (-1)^n C_{2n;p} \langle b^2 \rangle^n [\hat{G}(p)]^{2n} , \quad (7.7)$$

$$\hat{J}(p) = \sum_{n=2}^{\infty} \sum_p^J (-1)^n C_{2n;p} \langle b^2 \rangle^n [\hat{G}(p)]^{2n-1} . \quad (7.8)$$

Let us write

$$\hat{J}(p) = \hat{J}^{(1)}(p) + \hat{J}^{(2)}(p) + \hat{J}^{(3)}(p) , \quad (7.9)$$

where $\hat{J}^{(1)}(p)$, $\hat{J}^{(2)}(p)$, and $\hat{J}^{(3)}(p)$ are the contributions associated with Figs. 12a, 12b, and 12c, respectively. It can be seen with the aid of the diagrams that the terms of (7.8) included in $\hat{J}^{(1)}(p)$ are in one-to-one correspondence to the totality of terms in the expansion

$$[\hat{H}(p)]^2 = \sum_{n,m} \sum_{p,q}^{irr} (-1)^{n+m} C_{2n;p} C_{2m;q} \langle b^2 \rangle^{n+m} [\hat{G}(p)]^{2n+2m} , \quad (7.10)$$

which follows from (7.7). Evaluating the corresponding terms by means of the rule $C_{2n;p} = a^{n-1}$ (all nonvanishing irreducible C's) and then summing, we readily find

$$\hat{J}^{(1)}(p) = [\hat{G}(p)]^{-1} [\hat{H}(p)]^2. \quad (7.11)$$

A similar correspondence exists for $\hat{J}^{(2)}(p)$, and we thereby find

$$\hat{J}^{(2)}(p) = a [\hat{G}(p)]^{-1} [\hat{H}(p)]^2. \quad (7.12)$$

The evaluation of $\hat{J}^{(3)}(p)$ is somewhat more involved. Let us write

$$\hat{J}^{(3)}(p) = \sum_{i=1}^{\infty} \hat{J}^{(3,i)}(p), \quad (7.13)$$

where $\hat{J}^{(3,i)}(p)$ is the contribution from the i th consolidated diagram in the sequence of Fig. 12c. The series for $\hat{J}^{(3,1)}(p)$ is in one-to-one correspondence with that for $[\hat{H}(p)]^4$, and we find

$$\hat{J}^{(3,1)}(p) = -a^2 \langle b^2 \rangle^{-1} [\hat{G}(p)]^{-3} [\hat{H}(p)]^4 \quad (7.14)$$

by comparing corresponding terms. In a similar fashion, comparing the series for $\hat{J}^{(3,i+1)}(p)$ and $[\hat{H}(p)]^2 \hat{J}^{(3,i)}(p)$, we find

$$\hat{J}^{(3,i+1)}(p) = -a \langle b^2 \rangle^{-1} [\hat{G}(p)]^{-2} [\hat{H}(p)]^2 \hat{J}^{(3,i)}(p), \quad (7.15)$$

whence,

$$\hat{J}^{(3)}(p) = \hat{J}^{(3,1)}(p) - a \langle b^2 \rangle^{-1} [\hat{G}(p)]^{-2} [\hat{H}(p)]^2 \hat{J}^{(3)}(p). \quad (7.16)$$

Combining our results (and suppressing the argument p) we have, finally,

$$\hat{J} = \hat{G}^{-1} \hat{H}^2 \left[1 + \left(\frac{a}{1 + a \langle b^2 \rangle - \hat{G}^{-1} \hat{H}^2} \right) \right] . \quad (7.17)$$

Eliminating $\hat{H}$ and $\hat{J}$ from (7.6) and (7.17), we obtain

$$\langle b^2 \rangle \hat{G}^4 + p \langle b^2 \rangle + ap^2 \hat{G}^3 - (\langle b^2 \rangle + 3ap^2) \hat{G}^2 + 3ap \hat{G} - a = 0 . \quad (7.18)$$

The relevant branch of (7.18) is the one for which $\tilde{G}(\omega) = \pi^{-1} \text{Re} \{ \hat{G}(-i\omega) \}$ reduces to (5.6) when $a = 0$, and it is easily verified that such a branch exists. Let us examine the behavior of $\tilde{G}(\omega)$ on this branch as a is varied. For sufficiently small a , it is plausible to assume that our assignment of values to the C 's is realizable. As we increase a , we may plausibly anticipate that $\tilde{G}(\omega)$ will begin to violate (2.4), or at least will exhibit some discontinuous change in its dependence on a , when a critical value is reached for which the C 's are unrealizable by any assignment of values to the ϕ 's consistent with (4.4). This argument is supported by the example of Section 6.

The behavior of (7.18) is accessible by standard techniques for quartic equations. One finds that the form of $\tilde{G}(\omega)$ on the branch of interest varies continuously with a for $-1/4 < a < \infty$. Over this range,

$$\tilde{G}(0) = (\sqrt{2}\pi b_*)^{-1} \left[1 + (1 + 4a)^{1/2} \right]^{1/2} , \quad (7.19)$$

and, for $a > 0$, $\tilde{G}(\omega)$ decreases monotonically with increase of ω^2 up to a cut-off frequency given by

$$\omega_c^2 = \frac{1}{2} a^{-1} s^{-2} (1 + s)^2 \left[1 - (1 - 4as)^{1/2} \right] \langle b^2 \rangle , \quad (7.20)$$

where

$$s = 2 - (1 + a^{-1} + D^{1/2})^{1/3} - (1 + a^{-1} - D^{1/2})^{1/3} ,$$

$$D = (1 + a^{-1})^2 - (1 - \frac{2}{3}a^{-1})^3 .$$

As a increases, ω_c^2 increases monotonically. For $a < -1/4$, $\tilde{G}(0)$ is complex, and we conclude that this gives a lower bound to the range of realizability. There appears to be no upper bound, at least on the basis of the present considerations.

The properties just listed appear to justify the tentative conclusion that the present model is realizable for $a = 1$. It should be emphasized, however, that the arguments given do not constitute a proof of realizability. The latter would require an explicit prescription for constructing ϕ 's which yield the model. We have not found such a prescription, and therefore we regard the investigation of the present model as incomplete.

The function $\tilde{G}(\omega)$ for $a = 1$ is compared in Fig. 13 with (5.6) and (2.7). It will be noted that the present model gives a close approximation to (2.7) and represents a substantial improvement over the random coupling model.

The apparent realizability of the present model, and the significant improvement it gives over the random coupling model, suggest that there may be an infinite sequence of closed, realizable stochastic models in which successively broader classes of irreducible C 's are given the value 1 and such that $G(t)$ converges rapidly to its value in the true problem. If so, the coupling coefficients $A_{[n,r,s]}$ in (4.2) will exhibit a distribution, as functions of n , r , and s , which clusters more and more closely about the diagonal values $A_{[n,r,s]} = \delta_{r,n} \delta_{s,n}$ as one ascends the sequence. Thus we may hope that any given

dynamical properties of the models should converge in a statistical sense to those of the original collection of uncoupled oscillators. The analytical complexity of models higher than the two already described is formidable.

8. DRIVEN RANDOM OSCILLATOR

Let us add to the right side of (4.2) a forcing term $f_{[n]}(t)$ which is identically distributed for each n , statistically independent for different n , and statistically independent of $b_{[r]}$ for all n and r . Let us take the initial conditions

$$q_{[n]}(t_0) = 0, \quad f_{[n]}(t_0) = 0. \quad (8.1)$$

Writing

$$f_{[n]}(t) = \bar{f}(t) + f'_{[n]}(t), \quad \bar{f}(t) = \langle f_{[n]}(t) \rangle, \quad (8.2)$$

and introducing collective forces $f_\alpha(t)$ defined in correspondence to (3.1), we find

$$\begin{aligned} \langle M^{-1/2} f_0(t) \rangle &= \bar{f}(t), & \langle |M^{-1/2} f_0(t) - \bar{f}(t)|^2 \rangle &= O(M^{-1}), \\ \langle f_\alpha(t) \rangle &= 0, & \langle f_\alpha(t) f_\beta^*(t') \rangle &= \delta_{\alpha,\beta} F(t, t') \quad (\alpha \neq 0), \end{aligned} \quad (8.3)$$

where

$$F(t, t') = \langle f_{[n]}(t) f_{[n]}^*(t') \rangle.$$

We see that $f_0(t)$ plays a special role: When $M \rightarrow \infty$, the quantity $M^{-1/2} f_0(t) = M^{-1} \sum_n f_{[n]}(t)$ takes the sharp value $\bar{f}(t)$. It is easily verified

that $f_{\alpha}(t)$ depends only on the $f_{[n]}(t)$ if $\alpha \neq 0$.

In place of (4.1) we now have

$$dq_{\alpha}(t)/dt = -iM^{-1/2} \sum_{\beta} \phi_{\alpha, \beta, \alpha-\beta} b_{\beta} q_{\alpha-\beta}(t) = f_{\alpha}(t) . \quad (8.4)$$

Suppose that (8.4) is formally integrated from t_0 and solved by iteration. By arguments similar to those which give (4.7) we find

$$\langle q_{\alpha}(t) \rangle = 0 \quad (\alpha \neq 0) , \quad \langle q_{\alpha}(t) q_{\gamma}^*(t') \rangle = 0 \quad (\alpha \neq \gamma) . \quad (8.5)$$

By (8.5) and (3.3) we have

$$\langle q_{[n]}(t) \rangle = \bar{q}(t) , \quad \bar{q}(t) = \langle M^{-1/2} q_0(t) \rangle . \quad (8.6)$$

It may be verified from a term-by-term examination of the iteration solution that the variance of $M^{-1/2} q_0(t) = M^{-1} \sum_n q_{[n]}(t)$ is $O(M^{-1})$. Thus, $M^{-1/2} q_0(t)$ is statistically sharp ($M \rightarrow \infty$) and may be identified with $\bar{q}(t)$.¹⁹

Because of the special role played by $q_0(t)$, it is convenient to impose, in addition to (4.4) and (4.13), the condition²⁰

$$\phi_{\mu, \lambda, \sigma} = 1 \quad (\mu \text{ or } \sigma = 0) . \quad (8.7)$$

Then from (8.4) we obtain ($M \rightarrow \infty$)

$$d\bar{q}(t)/dt + iM^{-1} \sum_{\beta} \langle b_{-\beta} q_{\beta}(t) \rangle = \bar{f}(t) , \quad (8.8)$$

$$dq_{\alpha}(t)/dt = -ib_{\alpha} \bar{q}(t) - iM^{-1/2} \sum_{\beta} \phi_{\alpha, \alpha-\beta, \beta} b_{\alpha-\beta} q_{\beta}(t) + f_{\alpha}(t) \quad (\alpha \neq 0) , \quad (8.9)$$

where $\sum_{\beta}^{\prime}$ implies that $\beta = 0$ is excluded.

Let us write

$$q_{[n]}(t) = \bar{q}(t) + q_{[n]}'(t), \quad (8.10)$$

where, by (8.6), $\langle q_{[n]}'(t) \rangle = 0$. We shall call $\bar{q}(t)$ and $q_{[n]}'(t)$ the 'coherent' and 'incoherent' amplitudes, respectively. An explicit solution for $\bar{q}(t)$ is readily obtained. From the definition of the response matrix, and the statistical independence of the f 's and the b 's, we have

$$\langle q_0(t) \rangle = \sum_a \int_{t_0}^t \langle G_{0,a}(t-s) f_a(s) \rangle ds = \sum_a \int_{t_0}^t \langle G_{0,a}(t-s) \rangle \langle f_a(s) \rangle ds,$$

whence, by (8.3), (8.6), and (4.9), we obtain

$$\bar{q}(t) = \int_{t_0}^t G(t-s) \bar{f}(s) ds. \quad (8.11)$$

Consider now the covariance of the incoherent amplitude. Let

$$Q_{[n,m]}(t, t') = q_{[n]}'(t) q_{[m]}'^*(t'), \quad Q_{\alpha, \gamma}(t, t') = q_{\alpha}(t) q_{\gamma}^*(t').$$

Using (3.3), (8.5), and (8.6), we find

$$\langle Q_{[n,m]}(t, t') \rangle = M^{-1} \sum_a \exp[-i2\pi(n-m)a/M] \langle Q_{\alpha, \alpha}(t, t') \rangle + O(M^{-1}). \quad (8.12)$$

Therefore, if $\langle Q_{\alpha, \alpha}(t, t') \rangle$ is independent of α ($\alpha \neq 0$), we have ($M \rightarrow \infty$)

$$\langle Q_{[n,m]}(t, t') \rangle = \delta_{n,m} Q(t, t'),$$

$$Q(t, t') = \langle Q_{\alpha, \alpha}(t, t') \rangle \quad (\alpha \neq 0). \quad (8.13)$$

Let us assume hereafter that (8.13) holds. As we shall see shortly, this will be the case when (4.13) and (8.7) are satisfied.

An important statistical property is

$$\langle b_{\beta} b_{-\beta} \dots b_{\mu} b_{-\mu} Q_{\alpha, \alpha}(t, t') \rangle = \langle b^2 \rangle \dots \langle b^2 \rangle Q(t, t') = 0 (M^{-1}) \quad (8.14)$$

$$(\alpha \neq 0, \quad |\alpha| \neq |\beta| \neq \dots \neq |\mu|) .$$

For the case of uncoupled oscillators (all ϕ 's = 1) this follows directly from (3.1) and the statistical independence of $b_{[n]}, q_{[n]}(t)$ and $b_{[r]}, q_{[r]}(t)$ for $n \neq r$. In the general case, it may be verified for each term of the iteration expansion of the left side of (8.14).¹⁹

From (8.9) we have

$$\partial Q(t, t') / \partial t = S(t, t') + S_C(t, t') + S_F(t, t') , \quad (8.15)$$

where

$$S(t, t') = \sum_{\beta} \langle S_{\alpha, \alpha-\beta, \beta}(t, t') \rangle \quad (\alpha \neq 0) ,$$

$$S_{\alpha, \alpha-\beta, \beta}(t, t') = -iM^{-1/2} \phi_{\alpha, \alpha-\beta, \beta} b_{\alpha-\beta} q_{\beta}(t) q_{\alpha}^*(t') , \quad (8.16)$$

$$S_C(t, t') = -i \langle b_{\alpha} q_{\alpha}^*(t') \rangle \bar{q}(t) \quad (\alpha \neq 0) , \quad (8.17)$$

$$S_F(t, t') = \langle q_{\alpha}^*(t') f_{\alpha}(t) \rangle \quad (\alpha \neq 0) . \quad (8.18)$$

Our notation anticipates the fact that the expressions given for $S(t, t')$, $S_C(t, t')$, and $S_F(t, t')$ are individually independent of α . It should be noted that the similar equation for $\partial Q(t, t') / \partial t'$ is redundant with (8.15) because of

the property

$$Q(t, t') = Q^*(t', t), \quad (8.19)$$

which follows from the definition of $Q(t, t')$.

The quantity $S_F(t, t')$ is readily evaluated by an argument similar to that which gave (8.11). We have

$$\langle q_a^*(t') f_a(t) \rangle = \sum_{\beta} \int_{t_0}^{t'} \langle G_{a,\beta}^*(t'-s) \rangle \langle f_{\beta}^*(s) f_a(t) \rangle ds,$$

whence

$$S_F(t, t') = \int_{t_0}^{t'} G^*(t'-s) F(t, s) ds. \quad (8.20)$$

The evaluation of $\langle S_{a,\alpha-\beta,\beta}(t, t') \rangle$ parallels that of $\langle H_{a-\beta,\beta,a}(t) \rangle$ in Section 4. We expand $q_{\beta}(t)$ and $q_a^*(t')$ by iteration of the integrated form of (8.9), leaving $\bar{q}(t)$ (which is known) explicitly in the expansion. Then we average and note the regularities imposed by the sum rule for indices and the statistical properties of the b 's and the f 's. The expansion for $q_a^*(t')$ involves factors ϕ^* . If we express these as ϕ factors by (4.4), we are led, eventually, to the irreducible diagram expansion

$$\langle S_{a,\alpha-\beta,\beta}(t, t') \rangle = M^{-1} \sum_{n=1}^{\infty} \sum_p^{irr} C_{2n,p}(\alpha, \alpha-\beta, \beta) \xi_{2n,p}(t, t') \quad (8.21)$$

$$(\alpha, \beta \neq 0, \quad |\alpha| \neq |\beta| \neq |\alpha-\beta|),$$

where the $\xi_{2n,p}(t, t')$ are independent of α and β . From this we obtain²¹

$$S(t, t') = \sum_{n=1}^{\infty} \sum_p^{irr} C_{2n,p} \xi_{2n,p}(t, t'). \quad (8.22)$$

The $\xi_{2n;p}(t, t')$ may be determined by the variational method used for the $\xi_{2n;p}(t)$ in Section 4. For variations which leave the $C_{2n;p}$ unchanged, we have, in correspondence to (4.16),

$$\Delta \langle S_{\alpha, \alpha-\beta, \beta}(t, t') \rangle = M^{-1} \sum_n \sum_p^{\text{irr}} \xi_{2n;p}(t, t') \Delta C_{2n;p}(\alpha, \alpha-\beta, \beta). \quad (8.23)$$

The variation (4.17), with the notation change $\beta \rightarrow \alpha-\beta$, produces additional terms on the right sides of the equations of motion for $q_a^*(t')$ and $q_\beta(t)$. In correspondence to (4.19), we find (to order $M^{-1/2}$)

$$\begin{aligned} \Delta q_a^*(t') &= \int_{t_0}^{t'} G_{\alpha, \alpha}^*(t'-s) \left[iM^{-1/2} \Delta \phi_{\alpha, \alpha-\beta, \beta}^{b_{\beta-\alpha} q_\beta^*(s)} \right] ds, \\ \Delta q_\beta(t) &= \int_{t_0}^t G_{\beta, \beta}(t-s) \left[-iM^{-1/2} \Delta \phi_{\beta, \beta-\alpha, \alpha}^{b_{\beta-\alpha} q_\alpha(s)} \right] ds. \end{aligned} \quad (8.24)$$

Then (to order M^{-1}) we find, noting $\Delta \phi_{\alpha, \alpha-\beta, \beta}^* = \Delta \phi_{\beta, \beta-\alpha, \alpha}$,

$$\begin{aligned} \Delta \langle S_{\alpha, \alpha-\beta, \beta}(t, t') \rangle &= M^{-1} \Delta C_{2;1}(\alpha, \alpha-\beta, \beta) \\ &\times \left[\int_{t_0}^{t'} \langle G_{\alpha, \alpha}^*(t'-s) b_{\alpha-\beta} b_{\beta-\alpha} q_\beta(t) q_\beta^*(s) \rangle ds \right. \\ &\quad \left. - \int_{t_0}^t \langle G_{\beta, \beta}(t-s) b_{\alpha-\beta} b_{\beta-\alpha} q_\alpha^*(t') q_\alpha(s) \rangle ds \right]. \end{aligned} \quad (8.25)$$

Using (8.14) and the sharpness of $G_{\alpha,\alpha}$ and $G_{\beta,\beta}$ to reduce the averages in the limit $M \rightarrow \infty$, we have, finally,

$$\xi_{2;1}(t, t') = \langle b^2 \rangle \left[\int_{t_0}^{t'} G^*(t'-s) Q(t, s) ds - \int_{t_0}^t G(t-s) Q^*(t', s) ds \right]. \quad (8.26)$$

It is noteworthy that this expression depends on the driving forces only implicitly, through their effect on $Q(t, t')$.

The higher $\xi_{2n;p}(t, t')$ may be found by introducing more general variations. The result is that $\xi_{2n;p}$ consists of a sum of terms each of which involves a $(2n-1)$ -fold time integration over a product of $2n-1$ factors G , one Q factor, and n factors $\langle b^2 \rangle$.

We have finally to evaluate $S_C(t, t')$. It can be shown from the iteration solution of (8.9) that $\langle b_{\alpha} q_{\alpha}^*(t') \rangle$, like $\langle Q_{\alpha,\alpha}(t, t') \rangle$, has an irreducible diagram expansion and is independent of α ($\alpha \neq 0$). From the latter fact we have

$$\langle b_{\alpha} q_{\alpha}^*(t') \rangle = M^{-1} \sum_{\beta} \langle b_{\beta} q_{\beta}^*(t') \rangle + O(M^{-1}).$$

Hence, in the limit $M \rightarrow \infty$ we obtain from (8.8) the result

$$S_C(t, t') = -\bar{q}(t) \left[d\bar{q}^*(t')/dt' - \bar{F}^*(t') \right]. \quad (8.27)$$

Let us now specialize to the random coupling model. By (5.3), we then have²⁰

$$S(t, t') = \xi_{2;1}(t, t'). \quad (8.28)$$

Eqs. (8.15), (8.19), (8.20), ^{(8.26),}(8.27), and (8.28) now permit the determination of $Q(t, t')$.

Twice the real part of (8.15) for $t = t'$ is the equation for the rate of change of the mean intensity $Q(t, t)$ of the incoherent oscillation. The quantities $2\text{Re} \{S_G(t, t)\}$ and $2\text{Re} \{S_F(t, t)\}$ represent contributions to $dQ(t, t)/dt$ from interaction with the coherent oscillation and incoherent driving forces respectively. From (8.26) and (8.28) we have

$$\text{Re} \{S(t, t)\} = 0. \quad (8.29)$$

Thus, using (8.27), we find

$$\frac{d}{dt} [Q(t, t) + \bar{q}(t)\bar{q}^*(t)] = 0, \quad (8.30)$$

when all the driving forces vanish. This consistency property, and (8.29) itself, are assured in advance because our equations constitute the exact description of a model for which (4.4) holds. It is also assured that the solution $Q(t, t')$ of our model equations obeys all the realizability conditions to which covariances are subject.²²

The simplest solution of (8.15) for the random coupling model results from taking forces which vanish except for impulses at $t = t_0$ such that $Q(t_0, t_0) = 1$, $\bar{q}(t_0) = 0$. Then it is easily seen from (8.26), and the property $G(t) = G^*(-t)$,²³ that (8.15) becomes identical with (5.4) (for $t, t' > t_0$) under the substitution

$$Q(t, t') = G(t - t'). \quad (8.31)$$

Thus (8.31) is the solution for random shock excitation, as is required for consistency. In general, $Q(t, t')$ and $G(t - t')$ do not have the same form.

9. NON-GAUSSIAN FREQUENCY DISTRIBUTION

We shall now describe briefly the generalizations required when the distribution of $b_{[n]}$ is non-Gaussian. The case $\langle b_{[n]} \rangle \neq 0$ can be treated by methods similar to those used for $\langle f_{[n]} \rangle \neq 0$ in Section 8. But it is simpler to maintain the condition $\langle b_{[n]} \rangle = 0$ and instead replace (8.4) by

$$(d/dt + i\bar{\omega} + \nu)q_a(t) + iM^{-1/2} \sum_{\beta} \phi_{a,\beta,a-\beta} b_{\beta} q_{a-\beta}(t) = f_a(t), \quad (9.1)$$

where $\bar{\omega}$ is the (real) mean frequency and we have also included a real damping factor ν . It is easy to see that this generalization implies only minor changes in our treatment if $b_{[n]}$ remains Gaussian. The factor $t^{2n}/2n!$ in the iteration series (4.14) is replaced by $G^{(0)}(*G^{(0)})^{2n}$, where

$$G^{(0)}(t) = \exp[-(i\bar{\omega} + \nu)t]. \quad (9.2)$$

However, the irreducible diagram expansions for $H(t)$ and $S(t,t')$, and the expressions for the $\zeta_{2n;p}(t)$ and the $\xi_{2n;p}(t,t')$, are unchanged in form. The effect of $\bar{\omega}$, $\nu \neq 0$ is implicitly expressed by the changed values of the functions G and Q which appear in these expressions. The only further changes in Sections 2-8 are the obvious replacements $d/dt \rightarrow (d/dt + i\bar{\omega} + \nu)$ and $\partial/\partial t \rightarrow (\partial/\partial t + i\bar{\omega} + \nu)$ where appropriate.

Now let us consider the general (non-Gaussian) case where the $b_{[n]}$ are identically distributed (with zero mean) for all n and statistically independent for different n . It is easy to verify from (3.1) and (3.2) that (3.8) remains valid. In place of (3.10), we find

$$\begin{aligned}
 \langle b_\alpha \rangle &= 0, & \langle b_\alpha b_\beta \rangle &= \delta_{\alpha+\beta} \langle b^2 \rangle, \\
 \langle b_\alpha b_\beta b_\gamma \rangle &= \delta_{\alpha+\beta+\gamma} M^{-1/2} \langle b^3 \rangle, \\
 \langle b_\alpha b_\beta b_\gamma b_\epsilon \rangle &= (\delta_{\alpha+\beta} \delta_{\gamma+\epsilon} + \delta_{\alpha+\gamma} \delta_{\beta+\epsilon} + \delta_{\alpha+\epsilon} \delta_{\beta+\gamma}) \langle b^2 \rangle^2 \\
 &\quad + \delta_{\alpha+\beta+\gamma+\epsilon} M^{-1} (\langle b^4 \rangle - 3 \langle b^2 \rangle^2), \\
 &\dots
 \end{aligned} \tag{9.3}$$

In general we find ($M \rightarrow \infty$) that all moments of the b_α with indices equal and opposite in pairs depend ^{only} on $\langle b^2 \rangle$ and have the same values as for a Gaussian distribution of $b_{[n]}$ with this variance. Moments for which the indices do not pair (we shall call them skew moments) have values which depend on the cumulants of the $b_{[n]}$ distribution; they tend individually to zero as $M \rightarrow \infty$.^{2h}

The presence of skew moments results in new classes of terms in (4.11) and consequently in (4.23). The simplest new term in (4.23) is

$$(-i)^3 D_{3;1} \langle b^3 \rangle G * G * G, \tag{9.4}$$

where

$$D_{3;1} = M^{-2} \sum_{\beta, \gamma} \theta_{\alpha, \beta, \alpha-\beta} \theta_{\alpha-\beta, \gamma, \alpha-\beta-\gamma} \theta_{\alpha-\beta-\gamma, -\beta-\gamma, \alpha} \tag{9.5}$$

It may be represented by Fig. 14a. The further terms represented by Figs. 14b and 14c are proportional to $\langle b^4 \rangle - 3 \langle b^2 \rangle^2$ and to $\langle b^2 \rangle \langle b^3 \rangle$, respectively.

It is possible to generalize our sequence of models so that closed equations are produced which systematically include more and more of the information expressed by the cumulants of the b distribution. We shall not attempt this

here. However, it is important to note that the equations for the random coupling model are identical to those already given no matter what the (zero-mean) b distribution may be. It is clear that $D_{3;1}$ vanishes for this model ($M \rightarrow \infty$), and it can be seen that all the higher new terms in (4.23) vanish also. The statistical properties of the random coupling model thus depend only on the variance $\langle b^2 \rangle$. Recalling (2.3), which is exact for the original uncoupled oscillators, we see that in certain respects the random coupling model actually will provide a better approximation for distributions of b which resemble (5.6) in form than for a Gaussian distribution.

In the physical analogs to the random oscillator which are our eventual interest, the distribution of the stochastic quantity corresponding to b may itself be determined by dynamical processes. In this case there may exist an alternative to the general treatment mentioned above. It may be physically reasonable to assume Gaussian initial conditions for the quantities corresponding to b and q . If the dynamical equations for these quantities are then treated as a simultaneous set, the non-Gaussian diagrams will not arise in any of the relevant sequence of models. We shall give an illustration at the end of Section 11.

10. PARTICLE IN A RANDOM POTENTIAL

Let the Schrödinger equation for a particle be²⁵

$$(\partial/\partial t - i\nabla^2) \psi(\underline{x}, t) = -iv(\underline{x}) \psi(\underline{x}, t), \quad (10.1)$$

where $v(\underline{x})$ is a real potential which is statistically distributed over an infinite ensemble of realizations of the system. This problem is an exact homolog to the random oscillator with respect to treatment by stochastic models.²⁶ Let us consider a collection of M systems such that the individual potentials

$v_{[n]}(x)$ are identically distributed for all n and are statistically independent for different n . Let $\psi_{[n]}(x, t)$ be the Schrödinger function for the n th system. Then we may define the collective quantities $\psi_a(x, t)$ and $v_a(x)$ in correspondence to (3.1) and consider the model equations

$$(\partial/\partial t - i\nabla^2) \psi_a(x, t) = -iM^{-1/2} \sum_{\beta} \phi_{a, \beta, a-\beta} v_{\beta}(x) \psi_{a-\beta}(x, t). \quad (10.2)$$

The $\phi_{a, \beta, a-\beta}$ will be identical quantities for corresponding models in the present problem and the random oscillator problem.

The condition (4.4) serves to maintain hermiticity in the present case. It is easily verified from (4.4) and (10.2) that the total probability

$$\sum_n \int |\psi_{[n]}(x, t)|^2 d^3x = \sum_a \int |\psi_a(x, t)|^2 d^3x$$

and the total energy

$$\int \left[- \sum_n \psi_{[n]}^* \nabla^2 \psi_{[n]} + \sum_{n, r, s} \psi_{[n]}^* A_{[n, r, s]} v_{[r]} \psi_{[s]} \right] d^3x,$$

where $A_{[n, r, s]}$ is defined by (4.3), are conserved. However, the individual quantities $\int |\psi_{[n]}|^2 d^3x$ are not constants of motion, in general. The systems in the collection exchange particles as well as energy.

Let us now take the case where $v_{[n]}(x)$ has a multi-variate Gaussian distribution. This implies that all odd-order moments vanish and that all even-order moments are expressible in terms of the covariance $V(x, x') = \langle v_{[n]}(x) v_{[n]}(x') \rangle$. In the collective representation we have

$$\langle v_a(x) v_\beta(x') \rangle = \delta_{a+\beta} V(x, x'),$$

$$\begin{aligned} \langle v_a(x) v_\beta(x') v_\gamma(x'') v_\varepsilon(x''') \rangle &= \delta_{a+\beta} \delta_{\gamma+\varepsilon} V(x, x') V(x'', x''') \\ &+ \delta_{a+\gamma} \delta_{\beta+\varepsilon} V(x, x'') V(x', x''') + \delta_{a+\varepsilon} \delta_{\beta+\gamma} V(x, x''') V(x', x''), \end{aligned} \quad (10.3)$$

The analog of (3.8) holds, of course, whatever the distribution.

Let us define the Green's function $G_{[n,m]}(x, t | x', t')$ as the solution (for all t) of the model equation for $\psi_{[n]}(x, t)$ under the initial condition $\psi_{[n]}(x, t') = \delta_{r,m} \delta(x - x')$, and make a corresponding definition for $G_{a,\gamma}(x, t | x', t')$. Then, in correspondence to the analysis in Section 4, we find, when (4.13) is satisfied,

$$\begin{aligned} (\partial/\partial t - i \nabla_x^2) G(x, t | x', t') &= \sum_{n=1}^{\infty} \sum_p^{irr} (-1)^n c_{2n,p} \zeta_{2n,p}(x, t | x', t'), \\ G(x, t' | x', t') &= \delta(x - x'), \end{aligned} \quad (10.4)$$

where

$$\begin{aligned} \langle G_{[n,m]}(x, t | x', t') \rangle &= \delta_{n,m} G(x, t | x', t'), \\ \langle G_{a,\gamma}(x, t | x', t') \rangle &= \delta_{a,\gamma} G(x, t | x', t'), \end{aligned} \quad (10.5)$$

and $\zeta_{2n,p}(x, t | x', t')$ is the homolog of $\zeta_{2n,p}(t - t')$.

The functions $\zeta_{2n,p}(x, t | x', t')$ may be determined by the variational method used in Section 4. The variation (4.17) produces, in correspondence to (4.19), the variation

$$\Delta G_{\alpha-\beta, \alpha}(\underline{x}, t | \underline{x}', t') = \int_{t'}^t ds \int d^3 y G_{\alpha-\beta, \alpha-\beta}(\underline{x}, t | \underline{y}, s) \\ \times \left[-iM^{-1/2} \Delta \phi_{\alpha-\beta, -\beta, \alpha} v_{-\beta}(\underline{y}) G_{\alpha, \alpha}(\underline{y}, s | \underline{x}', t') \right], \quad (10.6)$$

and we are led, thereby, to the result

$$\zeta_{2;1}(\underline{x}, t | \underline{x}', t') = \int_{t'}^t ds \int d^3 y v(\underline{x}, \underline{y}) G(\underline{x}, t | \underline{y}, s) G(\underline{y}, s | \underline{x}', t'), \quad (10.7)$$

which corresponds to (4.21). This result depends on the fact that $G_{\alpha, \alpha}(\underline{x}, t | \underline{x}', t')$ is statistically sharp ($M \rightarrow \infty$), which may be demonstrated in the same way as for $G_{\alpha, \alpha}(t)$ of Section 4.

The result for $\zeta_{4;3}(\underline{x}, t | \underline{x}', t')$ is

$$\zeta_{4;3}(\underline{x}, t | \underline{x}', t') = \int_{t'}^t ds_1 \int_{t'}^{s_1} ds_2 \int_{t'}^{s_2} ds_3 \iiint d^3 y_1 d^3 y_2 d^3 y_3 v(\underline{x}, \underline{y}_2) v(\underline{y}_1, \underline{y}_3) \\ \times G(\underline{x}, t | \underline{y}_1, s_1) G(\underline{y}_1, s_1 | \underline{y}_2, s_2) G(\underline{y}_2, s_2 | \underline{y}_3, s_3) G(\underline{y}_3, s_3 | \underline{x}', t'). \quad (10.8)$$

The structure of expressions (10.7) and (10.8) may be represented, as in Fig. 15, by an appropriate labelling of the vertices in the diagrams for $C_{2;1}$ and $C_{4;3}$. The expressions for all the higher $\zeta_{2n;p}(\underline{x}, t | \underline{x}', t')$ may be written down by analogy from the diagrams for the corresponding irreducible $C_{2n;p}$.

In the random coupling model, where $C_{2;1} = 1$ and all the higher irreducible $C_{2n;p}$ vanish, (10.4) becomes

$$\begin{aligned}
 (\partial/\partial t - i \nabla_{\underline{x}}^2) G(\underline{x}, t | \underline{x}', t') = \\
 - \int_{t'}^t ds \int d^3 y V(\underline{x}, \underline{y}) G(\underline{x}, t | \underline{y}, s) G(\underline{y}, s | \underline{x}', t') , \\
 G(\underline{x}, t' | \underline{x}', t') = \delta(\underline{x} - \underline{x}') . \quad (10.9)
 \end{aligned}$$

It should be pointed out that this result is independent of the assumption that the potential has a Gaussian distribution, provided that $\langle v_{[n]}(\underline{x}) \rangle = 0$ (cf. Section 9).²⁷

We are assured that the solutions of (10.9) will exhibit certain consistency properties because this is an exact equation for a realizable model. In particular, if (10.2) is transformed into the momentum representation, it follows from a straightforward extension of the arguments given in connection with (5.7) that $G(\underline{x}, t | \underline{x}', t')$ satisfies a basic spectral condition. The latter takes its simplest form for the homogeneous case $V(\underline{x}, \underline{x}') = V(\underline{x} - \underline{x}')$, in which $G(\underline{x}, t | \underline{x}', t')$ can depend only on $\underline{x} - \underline{x}'$ and $t - t'$. If we write

$$\begin{aligned}
 G_{\underline{k}}(t - t') &= \int d^3 y G(\underline{x}, t | \underline{x}', t') \exp(-i \underline{k} \cdot \underline{y}) \quad (\underline{y} = \underline{x} - \underline{x}') , \\
 \tilde{G}_{\underline{k}}(\omega) &= (2\pi)^{-1} \int_{-\infty}^{\infty} ds G_{\underline{k}}(s) \exp(i \omega s) , \quad (10.10)
 \end{aligned}$$

then the spectral condition is

$$\tilde{G}_{\underline{k}}(\omega) = |\tilde{G}_{\underline{k}}(\omega)| . \quad (10.11)$$

[We may note that $G_{\underline{k}}(t - t')$ is the diagonal response function for the amplitude in the mode $\underline{k}$.] Eq. (10.11) implies the reciprocity relation

$$G(\underline{x}, t | \underline{x}', t') = G^*(\underline{x}', t' | \underline{x}, t) , \quad (10.12)$$

also
which may be deduced directly from (10.2) and (4.4) in the general inhomogeneous case.

When $V(\underline{x}, \underline{x}') = V(\underline{x} - \underline{x}')$, (10.9) has the transform

$$(\partial/\partial t + ik^2)G_{\underline{k}}(t) = - \int_0^t ds \int d^3\underline{y} V_{\underline{k}-\underline{k}'} G_{\underline{k}'}(t-s) G_{\underline{k}}(s) ,$$

$$G_{\underline{k}}(0) = 1 , \quad (10.13)$$

where

$$V_{\underline{k}} = (2\pi)^{-3} \int d^3\underline{y} V(\underline{y}) \exp(-ik \cdot \underline{y}) .$$

It is possible to solve (10.13) easily for very high k (the WKBJ limit). This is of particular interest because it is well known that the perturbation approach breaks down in this limit. Let us take k sufficiently high that $V_{\underline{k}-\underline{k}'} \approx 0$ unless $|\underline{k}-\underline{k}'| \ll k$. Then it is plausible that we may replace $G_{\underline{k}'}(t-s)$ by $G_{\underline{k}}(t-s)$ in (10.13) and thereby obtain²⁸

$$(\partial/\partial t + ik^2)G_{\underline{k}}(t) = - \langle v^2 \rangle \int_0^t G_{\underline{k}}(t-s) G_{\underline{k}}(s) ds, \quad G_{\underline{k}}(0) = 1 , \quad (10.14)$$

where $\langle v^2 \rangle = V(0) = \int V_{\underline{k}} d^3k$.

Eq. (10.14) becomes identical with (5.4) under the transformation

$$G_{\underline{k}}(t) \rightarrow \exp(-ik^2 t) G(t) , \quad \langle v^2 \rangle \rightarrow \langle b^2 \rangle .$$

Consequently, we have

$$\tilde{G}_{\underline{k}}(\omega) = \begin{cases} \langle \pi v_* \rangle^{-1} \left[1 - \left(\frac{\omega - k^2}{2v_*} \right)^2 \right]^{1/2} & (|\omega - k^2| \leq 2v_*) \\ 0 & (|\omega - k^2| > 2v_*) , \end{cases} \quad (10.15)$$

where $v_* = \langle v^2 \rangle^{1/2}$. This correspondence between the WKBJ limit and the random oscillator is not confined to the random coupling model. If (10.4) is written in the k representation in this limit, and the previous transformation is made, the resulting equation is identical with (4.23). A particular consequence is that the WKBJ solution to the true problem (all ϕ 's = 1) is

$$\tilde{G}_k(\omega) = (2\pi \langle v^2 \rangle)^{-1/2} \exp \left[-1/2 (\omega - k^2)^2 / \langle v^2 \rangle \right], \quad (10.16)$$

in correspondence to (2.7). This result states that for sharp kinetic energy (sharp k) the total energy distribution follows the Gaussian potential energy distribution. Considered as an approximation to (10.16), the random coupling result (10.15) exhibits the qualitative physical fact that sharp momentum states are not sharp energy states. The quantitative form of (10.15) suggests that the random coupling model may represent a better approximation to the original problem if the potential distribution has a clipped rather than a Gaussian form (cf. Section 9). It should be noted that the cumulant-discard approximation scheme, when applied in the WKBJ limit, yields expressions for $\exp(ik^2 t) G_k(t)$ which are identical in form to (2.13). This implies discrete spectra $\tilde{G}_k(\omega)$, which is unphysical compared to the random coupling result.

The general correspondence between the WKBJ limit and the random oscillator includes, of course, the second stochastic model, discussed in Section 7. The WKBJ results for the two stochastic models and for the true problem in the Gaussian case are given by Fig. 13, if the horizontal and vertical axes are relabelled $(\omega - k^2)/v_*$ and $\pi v_* \tilde{G}_k(\omega)$, respectively. Away from the WKBJ limit, the analysis for the second model is considerably more difficult than for the random oscillator, although the same in principle. The equations are not reducible to

algebraic form, and the analogs to the inverses $[\hat{G}(p)]^{-1}$ and $\langle b^2 \rangle^{-1}$, which appeared in Section 7, must be defined by integral equations.

Let us assume that the Schrödinger fields are switched on at $t = t_0$ in such a way that the $\psi_{[n]}(\underline{x}, t_0)$ are statistically independent and identically distributed for all n and statistically independent of $v_{[r]}(\underline{x})$ for all r . Let

$$\psi_{[n]}(\underline{x}, t) = \bar{\psi}(\underline{x}, t) + \psi'^{[n]}(\underline{x}, t), \quad \bar{\psi}(\underline{x}, t) = \langle \psi_{[n]}(\underline{x}, t) \rangle. \quad (10.17)$$

We shall call $\bar{\psi}(\underline{x}, t)$ the coherent wave and $\psi'^{[n]}(\underline{x}, t)$ the incoherent wave. The evolution of the coherent amplitude and the incoherent covariance may be determined by direct correspondence to the analysis in Section 8.

Noting that our switch-on is equivalent to the action of impulsive sources $f_{[n]}(\underline{x}, t) = \psi_{[n]}(\underline{x}, t_0) \delta(t - t_0)$, we have, in correspondence to (8.11),

$$\bar{\psi}(\underline{x}, t) = \int G(\underline{x}, t | \underline{y}, t_0) \bar{\psi}(\underline{y}, t_0) d^3 y. \quad (10.18)$$

When the potential is statistically homogeneous, (10.18) has the transform

$$\bar{\psi}_{\underline{k}}(t) = G_{\underline{k}}(t - t_0) \bar{\psi}_{\underline{k}}(t_0), \quad (10.19)$$

where

$$\bar{\psi}_{\underline{k}}(t) = (2\pi)^{-3} \int \bar{\psi}(\underline{x}, t) \exp(-i \underline{k} \cdot \underline{x}) d^3 x.$$

In this case the various momentum modes of the coherent wave evolve independently. As our WKB limit results illustrate, $G_{\underline{k}}(t)$ has a continuous spectrum and therefore vanishes as $t \rightarrow \infty$. Consequently, the coherent wave eventually is extinguished by its interaction with the random potential.

In direct correspondence to the results obtained in Section 8, we have

$$\begin{aligned} \langle \psi_{[n]}^{\prime}(\underline{x}, t) \psi_{[m]}^{\prime*}(\underline{x}', t') \rangle &= \delta_{n,m} \mathbb{Y}(\underline{x}, t; \underline{x}', t'), \\ \langle \psi_{\alpha}(\underline{x}, t) \psi_{\gamma}^*(\underline{x}', t') \rangle &= \delta_{\alpha, \gamma} \mathbb{Y}(\underline{x}, t; \underline{x}', t') \quad (\alpha \neq 0), \end{aligned} \quad (10.20)$$

where $\mathbb{Y}(\underline{x}, t; \underline{x}', t')$ has the symmetry property

$$\mathbb{Y}(\underline{x}, t; \underline{x}', t') = \mathbb{Y}^*(\underline{x}', t'; \underline{x}, t) \quad (10.21)$$

and obeys

$$(\partial/\partial t - i\nabla_{\underline{x}}^2) \mathbb{Y}(\underline{x}, t; \underline{x}', t') = S(\underline{x}, t; \underline{x}', t') + S_C(\underline{x}, t; \underline{x}', t'), \quad (10.22)$$

with

$$S(\underline{x}, t; \underline{x}', t') = \sum_{n=1}^{\infty} \sum_p^{\text{irr}} C_{2n;p} \xi_{2n;p}(\underline{x}, t; \underline{x}', t') \quad (10.23)$$

and

$$S_C(\underline{x}, t; \underline{x}', t') = -\bar{\psi}(\underline{x}, t) (\partial/\partial t' + i\nabla_{\underline{x}'}^2) \bar{\psi}^*(\underline{x}', t'). \quad (10.24)$$

There is no term corresponding to $S_F(t, t')$ because we have not admitted sources for $t > t_0$.

The functions $\xi_{2n;p}(\underline{x}, t; \underline{x}', t')$, which are homologous to the $\xi_{2n;p}(t, t')$ in (8.22), may be determined by employing our variational procedure and noting the statistical property

$$\begin{aligned} \langle \psi_{\beta}(\underline{x}_1) \psi_{-\beta}(\underline{x}'_1) \dots \psi_{\mu}(\underline{x}_r) \psi_{-\mu}(\underline{x}'_r) \psi_{\alpha}(\underline{y}, t) \psi_{\alpha}^*(\underline{y}', t') \rangle \\ = V(\underline{x}_1, \underline{x}'_1) \dots V(\underline{x}_r, \underline{x}'_r) \mathbb{Y}(\underline{y}, t; \underline{y}', t') + O(M^{-1}) \\ (\alpha \neq 0, \quad |\alpha| \neq |\beta| \neq \dots \neq |\mu|), \end{aligned} \quad (10.25)$$

which corresponds to (8.14). In particular, we find

$$\xi_{2,1}(\underline{x}, t; \underline{x}', t') = \int d^3 y \, V(\underline{x}, \underline{y}) \left[\int_{t_0}^{t'} ds \, G^*(\underline{x}', t' | \underline{y}, s) \Psi(\underline{x}, t; \underline{y}, s) - \int_{t_0}^t ds \, G(\underline{x}, t | \underline{y}, s) \Psi^*(\underline{x}', t'; \underline{y}, s) \right]. \quad (10.26)$$

For the random coupling model,

$$S(\underline{x}, t; \underline{x}', t') = \xi_{2,1}(\underline{x}, t; \underline{x}', t'). \quad (10.27)$$

and we have a closed set of equations which determine $\Psi(\underline{x}, t; \underline{x}', t')$ when the initial functions $\Psi(\underline{x}, t_0; \underline{x}', t_0)$ and $\bar{\Psi}(\underline{x}, t_0)$ are given. As was the case for $G(\underline{x}, t | \underline{x}', t')$, certain important consistency properties necessarily are exhibited by the solution $\Psi(\underline{x}, t; \underline{x}', t')$ for any realizable model. In particular, we are assured that $\Psi(\underline{x}, t; \underline{x}', t')$ satisfies all the realizability conditions to which a covariance is subject. In the homogeneous case, where the spatial dependence of $\Psi(\underline{x}, t; \underline{x}', t')$ involves only $\underline{x} - \underline{x}'$, we must have

$$\Psi_{\underline{k}}(t, t) = |\Psi_{\underline{k}}(t, t)|, \quad (10.28)$$

where

$$\Psi_{\underline{k}}(t, t') = (2\pi)^{-3} \int \Psi(\underline{x}, t; \underline{x}', t') \exp(-i \underline{k} \cdot \underline{y}) d^3 y \quad (\underline{y} = \underline{x} - \underline{x}').$$

When $t_0 \rightarrow -\infty$ and a statistically stationary state has been set up, so that

$\Psi_{\underline{k}}(t, t') = \Psi_{\underline{k}}(t - t')$, we have, further,

$$\tilde{\Psi}_{\underline{k}}(\omega) = |\tilde{\Psi}_{\underline{k}}(\omega)|, \quad (10.29)$$

where

$$\tilde{\gamma}_{\underline{k}}(\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} \gamma_{\underline{k}}(t) \exp(i\omega t) dt.$$

Twice the real part of (10.22) for $\underline{x}', t' = \underline{x}, t$ represents the continuity equation for the ensemble mean of the quantum-mechanical probability of finding a particle. The left side is the quantum-mechanical equivalent of the substantial derivative of the mean probability density $\gamma(\underline{x}, t; \underline{x}, t)$ in the incoherent wave. The corresponding quantity for the coherent wave is $-2\text{Re} \{S_G(\underline{x}, t; \underline{x}, t)\}$. It is clear from (10.27) and (10.26) that $\text{Re} \{S(\underline{x}, t; \underline{x}, t)\}$ vanishes. Consequently, the continuity equation simply states that a particle enters the incoherent wave as it leaves the coherent wave.

The vanishing of $\text{Re} \{S(\underline{x}, t; \underline{x}, t)\}$ expresses the fact that the direct effect of the potential on the particles is to change their momentum rather than their position. To illustrate this, let us take $\bar{\psi}(\underline{x}, t) = 0$ and assume that the potential and the incoherent wave are statistically homogeneous. Then from the Fourier transforms of (10.22), (10.27), and (10.26) we obtain

$$\begin{aligned} d\gamma_{\underline{k}}(t, t)/dt = 2\text{Re} \int_{t_0}^t ds \int d^3k' \gamma_{\underline{k}-\underline{k}'} \left[G_{\underline{k}}(s-t) \gamma_{\underline{k}'}(t, s) \right. \\ \left. - G_{\underline{k}'}(t-s) \gamma_{\underline{k}}(s, t) \right], \end{aligned} \quad (10.30)$$

after noting (10.12) and (10.21). The quantity $\gamma_{\underline{k}}(t, t)$ is the mean probability density for finding a particle with momentum $\underline{k}$. The right side of (10.30) therefore is the rate of transfer of particles to this momentum for all other momenta $\underline{k}'$. It is easily verified from (10.30) that $\int \gamma_{\underline{k}}(t, t) d^3k$ is a constant of motion.

Suppose that the fields have been switched on at $t_0 = -\infty$ in such fashion that a stationary state exists at time t . Using (10.11) and (10.29), the right side of (10.30) may be rewritten so that we have

$$0 = d\tilde{V}_{\underline{k}}(t,t)/dt = \int_{-\infty}^{\infty} d\omega d^3k' V_{\underline{k}-\underline{k}'} \left[\tilde{G}_{\underline{k}}(\omega) \tilde{V}_{\underline{k}'}(\omega) - \tilde{G}_{\underline{k}'}(\omega) \tilde{V}_{\underline{k}}(\omega) \right]. \quad (10.31)$$

We note that the right side of (10.31) is the difference of two terms each of which is positive.²⁹ The first represents an input of particles to mode $\underline{k}$ from other modes $\underline{k}'$ and the second represents an output to these other modes. If the excitation of mode $\underline{k}$ only were to be slowly increased by some outside agency, it is clear that the output term would increase in magnitude while the input term would be initially unaffected. Thus, the random coupling model exhibits a plausible tendency to restore statistical equilibrium.

It will be noted that (10.31) is satisfied in general if

$$\tilde{V}_{\underline{k}}(\omega) = f(\omega) \tilde{G}_{\underline{k}}(\omega) \quad (10.32)$$

where $f(\omega)$ is a function independent of $\underline{k}$. Now it can be seen from their definitions that $\tilde{G}_{\underline{k}}(\omega)$ is proportional to the density of eigenstates of energy ω available to a particle of momentum $\underline{k}$,³⁰ while $\tilde{V}_{\underline{k}}(\omega)$ is proportional to the occupation of such states by particles of this momentum. Thus (10.32) has the usual form of a single-particle equilibrium distribution law if $f(\omega)$ is a function of ω/θ (θ = temperature) appropriate to the statistics of the particle. In a later paper, we shall deduce distribution laws of this form directly from a condition of statistical equilibrium under small perturbations in the coupling among systems in a collection, without appealing to probability distributions in the space of the eigenstates (such as the grand canonical distribution).

11. TURBULENCE DYNAMICS

The problem of turbulence dynamics serves to illustrate the application of our methods to equations of motion which are nonlinear in the dynamic variables. In order to keep the formalism as simple as possible, we shall work here with the one-dimensional scalar analog to the Navier-Stokes equation proposed by Burgers.³¹ The treatment of the Navier-Stokes equation for an incompressible fluid, which we shall discuss briefly, does not differ in essentials.

Burgers' equation is

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2} \right) u(x, t) = - u(x, t) \frac{\partial u(x, t)}{\partial x} = - \frac{1}{2} \frac{\partial}{\partial x} [u(x, t)]^2. \quad (11.1)$$

The function $u(x, t)$ may be interpreted as the velocity of an infinitely compressible fluid, of constant kinematic viscosity ν , executing one-dimensional motion.

If $\nu = 0$, the quantities

$$\int_{-\infty}^{\infty} u(x, t) dx, \quad \frac{1}{2} \int_{-\infty}^{\infty} [u(x, t)]^2 dx$$

are both constants of motion. We shall call them 'momentum' and 'energy' respectively. [This is not their accurate meaning, however, on the basis of the interpretation just suggested for $u(x, t)$.]

If an infinitesimal forcing term $\delta f(x, t)$, is added to the right side of (11.1) for $t > t_0$, the response is

$$\delta u(x, t) = \int_{t_0}^t ds \int_{-\infty}^{\infty} dy G_{[\]}(x, t | y, s) \delta f(y, s),$$

where the infinitesimal Green's function $G_{[\square]}(x, t | x', t')$ obeys

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2} \right) G_{[\square]}(x, t | x', t') = - \frac{\partial}{\partial x} \left[u(x, t) G_{[\square]}(x, t | x', t') \right],$$

$$G_{[\square]}(x, t' | x', t') = \delta(x - x'). \quad (11.2)$$

In correspondence to the procedure followed in Sections 4, 8, and 10, let us take a collection of systems with velocity fields $u_{[n]}(x, t)$, and Green's functions $G_{[n, m]}(x, t | x', t')$, pass to the collective representation, and consider, instead of (11.1) and (11.2), model equations of the form

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2} \right) u_a(x, t) = - \frac{1}{2} M^{-1/2} \sum_{\beta} \phi_{a, \beta, a-\beta} \frac{\partial}{\partial x} \left[u_{\beta}(x, t) u_{a-\beta}(x, t) \right], \quad (11.3)$$

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2} \right) G_{a, \gamma}(x, t | x', t') = - M^{-1/2} \sum_{\beta} \phi_{a, \beta, a-\beta} \frac{\partial}{\partial x} \left[u_{\beta}(x, t) G_{a-\beta, \gamma}(x, t | x', t') \right],$$

$$G_{a, \gamma}(x, t' | x', t') = \delta_{a, \gamma} \delta(x - x'). \quad (11.4)$$

As before, the ϕ 's are independent of x and t and the same for all ensemble-realizations of the collection.

We shall now impose upon the ϕ 's the three conditions

$$\phi_{a, \beta, a-\beta} = \phi_{a, a-\beta, \beta},$$

$$\phi_{-a, -\beta, -a+\beta} = \phi_{a, \beta, a-\beta}^*, \quad \phi_{a-\beta, -\beta, a} = \phi_{a, \beta, a-\beta}^*. \quad (11.5)$$

The first is a symmetry convention. It does not restrict the dynamics.³² The second insures that (11.3) preserves the property

$$u_a(x, t) = u_{-a}^*(x, t)$$

and therefore the reality of the $u_{[n]}(x, t)$. The third is identical with (4.4). It insures that

$$\frac{1}{2} \sum_n \int_{-\infty}^{\infty} [u_{[n]}(x, t)]^2 dx = \frac{1}{2} \sum_a \int_{-\infty}^{\infty} |u_a(x, t)|^2 dx$$

is a constant of motion if $\nu = 0$. The property

$$\frac{d}{dt} \int_{-\infty}^{\infty} u_{[n]}(x, t) dx = 0$$

follows from (11.3) for any values of the ϕ 's, provided the $u_{[n]}(x, t)$ vanish at $x = \infty$. In correspondence to (8.7) we shall also require

$$\phi_{\mu, \lambda, \sigma} = 1 \quad (\mu, \lambda \text{ or } \sigma = 0) . \quad (11.6)$$

The additional conditions which the ϕ 's now satisfy imply only minor modifications in the diagrammatic representation introduced in Section 4. Let us associate with $\phi_{\mu, \lambda, \sigma}$ and $\phi_{\mu, \lambda, \sigma}^*$ the vertices shown in Fig. 1a and 1b respectively. Then the rules for associating diagrams with $C_{2n; p}(\alpha, \beta, \alpha - \beta)$ and $C_{2n; p}$ are identical with those given in Section 4 if dashed lines are replaced by solid lines everywhere.³³ We shall assume hereafter that (4.13) is satisfied.

Let us take

$$u_{[n]}(x, t_0) = \bar{u}(x, t_0) + u_{[n]}'(x, t_0) , \quad (11.7)$$

where the initial values $u_{[n]}'(x, t_0)$ are identically distributed, with zero mean, for each n and statistically independent for different n . In correspondence to (3.8), it then follows that the moments of the $u_a(x, t_0)$ vanish unless the sum of

indices is zero. Now suppose that (11.3) is solved by iteration. From this property of the initial value moments, and the combination rule for indices in (11.3), we find

$$\langle u_{\alpha}(x,t)u_{\beta}(x',t') \dots \rangle = 0 \quad (\alpha + \beta + \dots \neq 0). \quad (11.8)$$

Similarly, the iteration solution of (11.4) yields

$$\langle G_{\alpha,\gamma}(x,t|x',t') \rangle = 0 \quad (\alpha \neq \gamma) \quad (11.9)$$

It follows immediately from (11.8) that

$$\langle u_{[n]}(x,t) \rangle = \langle M^{-1/2}u_0(x,t) \rangle$$

for all n . In correspondence to the similar result cited in Section 8, it can be shown that $M^{-1/2}u_0(x,t)$ is a sharp quantity ($M \rightarrow \infty$). Let us write

$$u_{[n]}(x,t) = \bar{u}(x,t) + u_{[n]}'(x,t), \quad \bar{u}(x,t) = \langle u_{[n]}(x,t) \rangle. \quad (11.10)$$

We shall call $\bar{u}(x,t)$ and $u_{[n]}'(x,t)$ the mean and fluctuating fields respectively. Identifying $M^{-1/2}u_0(x,t)$ with $\bar{u}(x,t)$ in the limit $M \rightarrow \infty$, and noting (11.6), we may now rewrite (11.3) and (11.4) in the form

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2} \right) \bar{u}(x,t) + \bar{u}(x,t) \frac{\partial \bar{u}(x,t)}{\partial x} = - \frac{1}{2} M^{-1} \sum_{\alpha} \frac{\partial}{\partial x} \langle u_{\alpha}(x,t)u_{-\alpha}(x,t) \rangle, \quad (11.11)$$

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2} \right) u_{\alpha}(x,t) + \frac{\partial}{\partial x} [\bar{u}(x,t)u_{\alpha}(x,t)] =$$

$$- \frac{1}{2} M^{-1/2} \sum_{\beta} \phi_{\alpha,\beta,\alpha-\beta} \frac{\partial}{\partial x} [u_{\beta}(x,t)u_{\alpha-\beta}(x,t)] \quad (\alpha \neq 0), \quad (11.12)$$

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2} \right) G_{\alpha, \gamma}(x, t | x', t') + \frac{\partial}{\partial x} \left[\bar{u}(x, t) G_{\alpha, \gamma}(x, t | x', t') \right] =$$

$$- M^{-1/2} \sum_{\beta}^{\prime} \delta_{\alpha, \beta, \alpha-\beta} \frac{\partial}{\partial x} \left[u_{\beta}(x, t) G_{\alpha-\beta, \gamma}(x, t | x', t') \right],$$

$$G_{\alpha, \gamma}(x, t' | x', t') = \delta_{\alpha, \gamma} \delta(x-x'). \quad (11.13)$$

where $\sum_{\beta}^{\prime}$ implies that $\beta = 0$ is to be omitted and $\sum_{\beta}^{\prime\prime}$ implies that both $\beta = 0$ and $\alpha - \beta = 0$ are to be omitted. It should be noted that (11.13) has the same form for $\alpha \neq 0$ and $\alpha = 0$. Eqs. (11.11) and (11.12) are coupled equations which determine the evolution of the mean and fluctuating fields.

Now let us assume that the distribution of the initial values $u_{[n]}'(x, t_0)$ is multivariate Gaussian. It can then be shown from the iteration solutions of (11.12) and (11.13), using arguments similar to those in Sections 4 and 8, that $\langle u_{\alpha}(x, t) u_{-\alpha}(x', t') \rangle$ is independent of α ($\alpha \neq 0$) and that $\langle G_{\alpha, \alpha}(x, t | x', t') \rangle$ is independent of α (all α). Then it follows from (11.8) and (11.9) that

$$\langle u_{[n]}'(x, t) u_{[m]}'(x', t') \rangle = \delta_{n, m} U(x, t; x', t'),$$

$$U(x, t; x', t') = \langle u_{\alpha}(x, t) u_{-\alpha}(x', t') \rangle \quad (\alpha \neq 0),$$

$$\langle G_{[n, m]}(x, t | x', t') \rangle = \delta_{n, m} G(x, t | x', t'),$$

$$G(x, t | x', t') = \langle G_{\alpha, \alpha}(x, t | x', t') \rangle. \quad (11.14)$$

In correspondence to our previous results, $G_{\alpha, \alpha}(x, t | x', t')$ is statistically sharp ($M \rightarrow \infty$), and the covariances satisfy

$$\langle u_{\alpha}(x,t)u_{-\alpha}(x',t')u_{\beta}(y,s)u_{-\beta}(y',s') \dots \rangle =$$

$$U(x,t;x',t')U(y,s;y',s') \dots + O(M^{-1})$$

$$(\alpha, \beta, \dots \neq 0, |\alpha| \neq |\beta| \neq \dots) \quad (11.15)$$

[cf. (8.14)]. It follows from (11.14) that $U(x,t;x',t')$ has the symmetry property

$$U(x,t;x',t') = U(x',t';x,t). \quad (11.16)$$

From (11.12) and (11.13) we find that $U(x,t;x',t')$ and $G(x,t|x',t')$ satisfy equations of the form

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2} \right) U(x,t;x',t') + \frac{\partial}{\partial x} \left[\bar{u}(x,t)U(x,t;x',t') \right] = S(x,t;x',t'), \quad (11.17)$$

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2} \right) G(x,t|x',t') + \frac{\partial}{\partial x} \left[\bar{u}(x,t)G(x,t|x',t') \right] = H(x,t|x',t'),$$

$$G(x,t'|x',t') = \delta(x-x'), \quad (11.18)$$

where it follows from the iteration solutions, with Gaussian $u_{[n]}'(x,t_0)$, that $S(x,t;x',t')$ and $H(x,t|x',t')$ have the forms

$$S(x,t;x',t') = \sum_n \sum_p^{irr} C_{2n,p} \xi_{2n,p}(x,t;x',t') \quad (11.19)$$

and

$$H(x,t|x',t') = \sum_n \sum_p^{irr} C_{2n,p} \zeta_{2n,p}(x,t|x',t'). \quad (11.20)$$

To complete the set of equations, we may rewrite (11.11) in the form

$$\left(\frac{\partial}{\partial t} - v \frac{\partial^2}{\partial x^2} \right) \bar{u}(x, t) + \bar{u}(x, t) \frac{\partial \bar{u}(x, t)}{\partial x} = - \frac{1}{2} \frac{\partial}{\partial x} U(x, t; x, t) . \quad (11.21)$$

Eq. (11.21) is the balance equation for mean 'momentum' density and (11.17) for $x', t' = x, t$ is the balance equation for the mean 'energy' density in the fluctuating field.

The functions $\xi_{2n;p}(x, t; x', t')$ and $\zeta_{2n;p}(x, t | x', t')$ may be determined by the variational procedure of Section 4, using (11.15) and the statistical sharpness of $G_{a,a}(x, t | x', t')$. The results for $\xi_{2;1}(x, t; x', t')$ and $\zeta_{2;1}(x, t | x', t')$ are

$$\begin{aligned} \xi_{2;1}(x, t; x', t') &= \frac{1}{2} \frac{\partial}{\partial x} \int_{t_0}^{t'} ds \int_{-\infty}^{\infty} dy G(x', t' | y, s) \frac{\partial}{\partial y} [U(x, t; y, s)]^2 \\ &+ \frac{\partial}{\partial x} \int_{t_0}^t ds \int_{-\infty}^{\infty} dy G(x, t | y, s) \frac{\partial}{\partial y} [U(x, t; y, s) U(x', t'; y, s)] , \end{aligned} \quad (11.22)$$

$$\begin{aligned} \zeta_{2;1}(x, t | x', t') &= \frac{\partial}{\partial x} \int_{t_0}^t ds \int_{-\infty}^{\infty} dy G(x, t | y, s) \\ &\times \frac{\partial}{\partial y} [U(x, t; y, s) G(y, s | x', t')] . \end{aligned} \quad (11.23)$$

In general $\xi_{2n;p}(x, t; x', t')$ consists of a sum of terms each of which involves a $(2n-1)$ -fold space-time integration over a product of $2n-1$ factors G and $n+1$ factors U . The terms comprising $\zeta_{2n;p}(x, t | x', t')$ each involve a $(2n-1)$ -fold

integration over a product of $2n$ factors G and n factors U .

We shall illustrate the variational procedure in the present case by outlining the analysis for $\xi_{2,1}(x,t;x',t')$. In correspondence to (8.16), we may write, for the present problem,

$$S(x,t;x',t') = \sum_{\beta}'' \langle S_{\alpha,\beta,\alpha-\beta}(x,t;x',t') \rangle \quad (\alpha \neq 0),$$

$$S_{\alpha,\beta,\alpha-\beta}(x,t;x',t') = -\frac{1}{2} M^{-1/2} \phi_{\alpha,\beta,\alpha-\beta} \frac{\partial}{\partial x} \left[u_{\beta}(x,t) u_{\alpha-\beta}(x,t) u_{-\alpha}(x',t') \right]. \quad (11.24)$$

Then the iteration solution yields

$$\begin{aligned} \langle S_{\alpha,\beta,\alpha-\beta}(x,t;x',t') \rangle &= M^{-1} \sum_{n=1}^{\infty} \sum_p^{\text{irr}} G_{2n;p}(\alpha,\beta,\alpha-\beta) \xi_{2n;p}(x,t;x',t') \\ (\alpha,\beta,\alpha-\beta \neq 0) \quad &(|\alpha| \neq |\beta| \neq |\alpha-\beta|), \end{aligned} \quad (11.25)$$

which corresponds to (8.21), and is the basis for (11.19). Now consider the variation (4.17). By using (11.5) several times, we find

$$\begin{aligned} \Delta \phi_{\alpha-\beta,-\beta,\alpha} &= \Delta \phi_{\alpha-\beta,\alpha,-\beta} = \Delta \phi_{\beta,-\alpha+\beta,\alpha} = \Delta \phi_{\beta,\alpha,-\alpha+\beta} \\ &= \Delta \phi_{-\alpha,-\beta,-\alpha+\beta} = \Delta \phi_{-\alpha,-\alpha+\beta,-\beta}. \end{aligned}$$

Hence, recalling (11.3), we find

$$\Delta u_{\beta}(x,t) = \int_{t_0}^t ds \int_{-\infty}^{\infty} dy G_{\beta,\beta}(x,t|y,s) \left\{ -M^{-1/2} \Delta \phi_{\beta,\alpha,-\alpha+\beta} \frac{\partial}{\partial y} \left[u_{\alpha}(y,s) u_{-\alpha+\beta}(y,s) \right] \right\}, \quad (11.26)$$

to order $M^{-1/2}$, with expressions of the same type for $\Delta u_{\alpha\beta}(x,t)$ and $\Delta u_{\alpha}(x',t')$.³⁴ These results correspond to (8.24). It is important to note that the perturbation terms are $O(M^{-1/2})$ so that the infinitesimal response functions may correctly be used to find the induced variations. Now we may express $\Delta(S_{\alpha,\beta,\alpha-\beta}(x,t;x',t'))$ to $O(M^{-1})$ in correspondence to (8.25), reduce the averages by using (11.15) and the sharpness of the $G_{\alpha,\alpha}$, and appeal to the analog of (8.23). Thereby we obtain the result (11.22).

The random coupling model for the present problem is obtained by assigning the ϕ 's as in Section 5, but with the additional constraints (11.5) and (11.6). It is clear that these constraints do not affect (5.3) in the limit $M \rightarrow \infty$. Hence, we have

$$S(x,t;x',t') = \xi_{2,1}(x,t;x',t'), \quad H(x,t|x',t') = \zeta_{2,1}(x,t|x',t') \quad (11.27)$$

for this model. These relations, together with (11.16)-(11.18) and (11.21)-(11.23), form a closed set which determine $\bar{u}(x,t)$, $U(x,t;x',t')$, and $G(x,t|x',t')$ in terms of the initial functions $\bar{u}(x,t_0)$ and $U(x,t_0;x',t_0)$.

The most essential difference between the present equations and the analogous ones for the random potential problem given in Section 10 is that $G(x,t|x',t')$ is not independent of $U(x,t;x',t')$ and $\bar{u}(x,t)$ in the present case; all three quantities now must be determined simultaneously. A further consequence of the non-linearity is that $\bar{u}(x,t)$ does not have an expression analogous to (10.18). The Green's function $G(x,t|x',t')$ can only describe the propagation of infinitesimal disturbances $\delta\bar{u}(x',t')$. In general, $\bar{u}(x,t) \neq 0$ even if $\bar{u}(x,t_0) = 0$ everywhere.

The Navier-Stokes equation for the velocity $u(x,t)$ of an infinite incompressible fluid of kinematic viscosity ν may be written, after elimination of the

pressure term,³ in the tensor form

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2 \right) u_i(\underline{x}, t) = - \frac{1}{2} P_{imn}(\nabla) \left[u_m(\underline{x}, t) u_n(\underline{x}, t) \right], \quad (11.28)$$

where

$$P_{imn}(\nabla) = \left(\delta_{i,m} - \nabla^{-2} \frac{\partial^2}{\partial x_i \partial x_m} \right) \frac{\partial}{\partial x_n} + \left(\delta_{i,n} - \nabla^{-2} \frac{\partial^2}{\partial x_i \partial x_n} \right) \frac{\partial}{\partial x_m}$$

and

$$\nabla^{-2} f(\underline{x}) = \int |\underline{x} - \underline{y}|^{-1} f(\underline{y}) d^3 y$$

for any f . We may treat the incompressible turbulence problem in direct analogy to the foregoing analysis by taking a collection of flow systems with individual velocity fields $u_i^{[n]}(\underline{x}, t)$ and considering the model equation³⁵

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2 \right) u_i^a(\underline{x}, t) = -M^{-1/2} P_{imn}(\nabla) \sum_{\beta} \phi_{a,\beta,a-\beta} \left[u_n^{\beta}(\underline{x}, t) u_m^{a-\beta}(\underline{x}, t) \right], \quad (11.29)$$

where the $u_i^a(\underline{x}, t)$ are the collective velocity fields. Conditions (11.5) now serve to insure momentum conservation and (when $\nu = 0$) energy conservation.

The final equations for the random coupling model which result from (11.29) are similar to those for Burgers' equation, but more complicated. In the case of homogeneous turbulence, they take their simplest form when transformed to correspond to a representation of the velocity field by spatial Fourier modes. They are then identical with equations for homogeneous turbulence derived previously by a different method.³⁶ The earlier derivation exploited the fact that the Fourier amplitudes of a homogeneous field have statistical properties which

closely resemble those of the collective coordinates used in the present paper (cf. Section 3). Unlike the present approach, which involves no geometrical symmetry restrictions and which may be extended to bounded flows, the earlier treatment is valid only in the homogeneous case. An extensive discussion of the energy dynamics of the random coupling model is given in reference 36.

We wish, finally, to give a very brief discussion of turbulent convection, which will serve to illustrate a point raised at the end of Section 9. Let $\psi(\underline{x}, t)$ represent the zero-mean fluctuations in the concentration of marked particles carried along with an incompressible turbulent flow which obeys (11.28). Then $\psi(\underline{x}, t)$ satisfies

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2 \right) \psi(\underline{x}, t) = -u_1(\underline{x}, t) \frac{\partial \psi(\underline{x}, t)}{\partial x_1}, \quad (11.30)$$

where κ is the molecular diffusivity. If $\psi_{[n]}(\underline{x}, t)$ and $\psi_a(\underline{x}, t)$ represent, respectively, the individual and collective fields for a collection of flows, the model equation corresponding to (11.30) is

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2 \right) \psi_a(\underline{x}, t) = -M^{-1/2} \sum_{\beta} \phi_{a, \beta, a-\beta} u_1^{\beta}(\underline{x}, t) \frac{\partial \psi_{a-\beta}(\underline{x}, t)}{\partial x_1}, \quad (11.31)$$

where (11.5) and (11.6) are satisfied.

The random coupling model equations which result from (11.31), under Gaussian initial conditions of the form we have taken before, are

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \kappa \nabla_{\underline{x}}^2 + \bar{u}_1(\underline{x}, t) \frac{\partial}{\partial x_1} \right) G(\underline{x}, t | \underline{x}', t') \\ &= \int_{t'}^t ds \int d^3 y U_{1j}(\underline{x}, t; \underline{y}, s) \frac{\partial G(\underline{x}, t | \underline{y}, s)}{\partial x_1} \frac{\partial G(\underline{y}, s | \underline{x}', t')}{\partial y_j}, \end{aligned}$$

$$G(\underline{x}, t | \underline{x}', t') = \delta(\underline{x} - \underline{x}'), \quad (11.32)$$

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \kappa \nabla_{\underline{x}}^2 + \bar{u}_1(\underline{x}, t) \frac{\partial}{\partial x_1} \right) \Psi(\underline{x}, t; \underline{x}', t') \\ &= \frac{\partial}{\partial x_1} \int_{t_0}^{t'} ds \int d^3 y U_{1j}(\underline{x}, t; \underline{y}, s) G(\underline{x}', t' | \underline{y}, s) \frac{\partial \Psi(\underline{x}, t; \underline{y}, s)}{\partial y_j} \\ &+ \frac{\partial}{\partial x_1} \int_{t_0}^t ds \int d^3 y U_{1j}(\underline{x}, t; \underline{y}, s) G(\underline{x}, t | \underline{y}, s) \frac{\partial \Psi(\underline{x}', t'; \underline{y}, s)}{\partial y_j}, \end{aligned}$$

where

$$\bar{u}_1(\underline{x}, t) = \langle u_1^{[n]}(\underline{x}, t) \rangle, \quad U_{1j}(\underline{x}, t; \underline{x}', t') = \langle u_1^{[n]}(\underline{x}, t) u_j^{[n]}(\underline{x}', t') \rangle,$$

$$\Psi(\underline{x}, t; \underline{x}', t') = \langle \psi_n(\underline{x}, t) \psi_n(\underline{x}', t') \rangle,$$

and $G(\underline{x}, t | \underline{x}', t')$ is the mean diagonal Green's function for the concentration field.³⁷ We have assumed $\langle \psi^{[n]}(\underline{x}, t) \rangle = 0$, a condition which is preserved by the equations of motion. A detailed study of the consequences of these equations when the velocity field is statistically homogeneous has been made by Roberts,³⁸ who derives the equations for this case by methods related to those of reference 36. Another special case has been discussed by the present author.³⁹

In accord with the discussion in Section 9, the random coupling equations for turbulent convection involve only $\bar{u}_1(\underline{x}, t)$ and the covariance tensor $U_{1j}(\underline{x}, t; \underline{x}', t')$, regardless of the distribution of the fluctuating part of the velocity field. Suppose, now, we ask how the higher statistical structure of the velocity field can be incorporated in higher stochastic models for the convection problem. If this structure were known explicitly, we could, in principle, insert the associated cumulants in the non-Gaussian terms, of the type (9.4), which con-

tribute in the higher models. An alternative procedure is to assume Gaussian initial conditions for both the concentration field and the fluctuating velocity field and then treat (11.29) and (11.31) as a simultaneous set, making the ϕ 's identical in the two equations. The sequence of higher models for this problem would commence with that of Section 7, and the non-Gaussian diagrams would never arise. The assumption of Gaussian initial conditions often may be physically plausible, particularly if the flow has persisted long enough that the higher statistical structure of the velocity field is determined principally by the dynamics rather than by the cumulants of the initial distribution.

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APPENDIX A

The second of relations (3.12) may be written

$$\langle G_{a,a}(t) G_{a,a}^*(t) \rangle - \langle G_{a,a}(t) \rangle \langle G_{a,a}^*(t) \rangle = O(M^{-1}), \quad (A.1)$$

where we use $\langle G_{a,a}(t) \rangle = G(t)$. Let the left side of (A.1) be expanded by iteration of (4.6). In each term of the expansion of $G_{a,a}(t)$ or $G_{a,a}^*(t)$, the sum of indices of the b factors is zero [cf. argument leading to (4.7)]. Consequently, $\langle G_{a,a}(t) \rangle \langle G_{a,a}^*(t) \rangle$ consists of terms of the form

$$\begin{aligned} & (-i)^{r-s} M^{-(r+s)/2} \sum_{\beta, \dots, \mu, \gamma, \dots, \sigma} (\text{product of } \phi's) \\ & \times \langle b_{\beta} \dots b_{\mu} b_{-\beta-\dots-\mu} \rangle \langle b_{\gamma} \dots b_{\sigma} b_{-\gamma-\dots-\sigma} \rangle t^{r+s}/r!s!, \end{aligned} \quad (A.2)$$

where there are r factors b in the first average and s in the second. For each such term there will be a corresponding term

$$\begin{aligned} & (-i)^{r-s} M^{-(r+s)/2} \sum_{\beta, \dots, \mu, \gamma, \dots, \sigma} (\text{product of } \phi's) \\ & \times \langle b_{\beta} \dots b_{\mu} b_{-\beta-\dots-\mu} b_{\gamma} \dots b_{\sigma} b_{-\gamma-\dots-\sigma} \rangle t^{r+s}/r!s! \end{aligned} \quad (A.3)$$

in the expansion of $\langle G_{a,a}(t) G_{a,a}^*(t) \rangle$, where the product of ϕ 's is identical for given indices $\beta, \dots, \gamma, \dots$. Let the ϕ 's be bounded. Then the difference of (A.2) and (A.3) is bounded in magnitude by

$$\begin{aligned} & M^{-(r+s)/2} |(\text{product of } \phi's)|_{\max} \sum_{\beta, \dots, \mu, \gamma, \dots, \sigma} \\ & \times |\langle b_{\beta} \dots b_{\mu} b_{-\beta-\dots-\mu} b_{\gamma} \dots b_{\sigma} b_{-\gamma-\dots-\sigma} \rangle - \langle b_{\beta} \dots b_{\mu} b_{-\beta-\dots-\mu} \rangle \langle b_{\gamma} \dots b_{\sigma} b_{-\gamma-\dots-\sigma} \rangle| \\ & \times t^{r+s}/r!s!. \end{aligned} \quad (A.4)$$

It now follows straightforwardly from (9.3) (we take the general non-Gaussian case) that (A.4) is $O(M^{-1})$ if $[(\text{product of } \phi\text{'s})]_{\max}$ is independent of M ($M \rightarrow \infty$). Similar analysis establishes (8.14) and (11.15) if the iteration solutions of (8.9) and (11.12), respectively, are used to express the equations in terms of the parameters and initial values, whose statistical properties are prescribed. In these solutions it is convenient to let the mean amplitudes remain in the expansions as parameters.

The significance of (3.12) was discussed in the text. Eqs. (8.14) and also (11.15) may be understood qualitatively as consequences of the fact that the dynamical behavior of a collective degree of freedom is determined ($M \rightarrow \infty$) by interaction with an infinite number of other degrees of freedom. The dynamical coupling with any given few of the other degrees of freedom is infinitesimal in the limit, and this implies a corresponding weakness of statistical dependence.

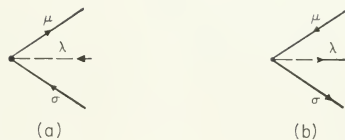


FIG. 1. VERTICES REPRESENTING $\phi_{\mu, \lambda, \sigma}$ AND $\phi_{\mu, \lambda, \sigma}^*$ FOR THE RANDOM OSCILLATOR

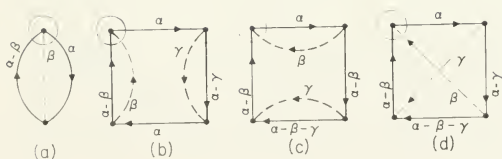


FIG. 2. DIAGRAMS FOR $C_{2,1}()$, $C_{4,1}()$, $C_{4,2}()$, AND $C_{4,3}()$

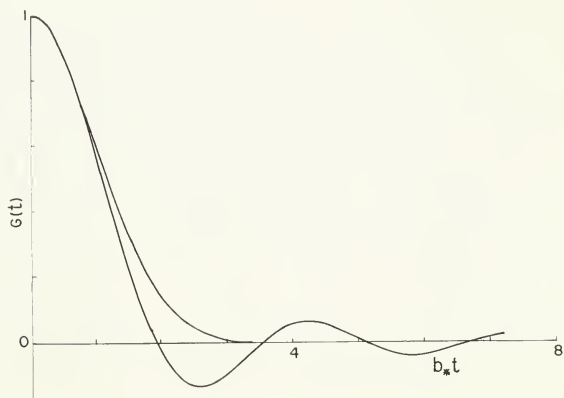


FIG. 3. COMPARISON OF $G(t)$ FOR THE TRUE PROBLEM AND THE RANDOM COUPLING MODEL

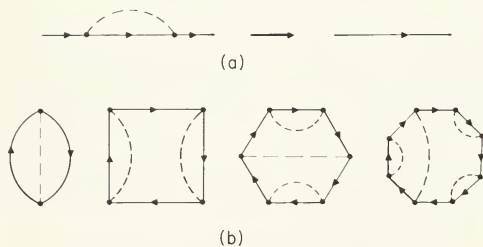


FIG. 4. (a) CONTRACTION OPERATION FOR THE RANDOM COUPLING MODEL

(b) TYPICAL DIAGRAMS INCLUDED IN THE RANDOM COUPLING MODEL

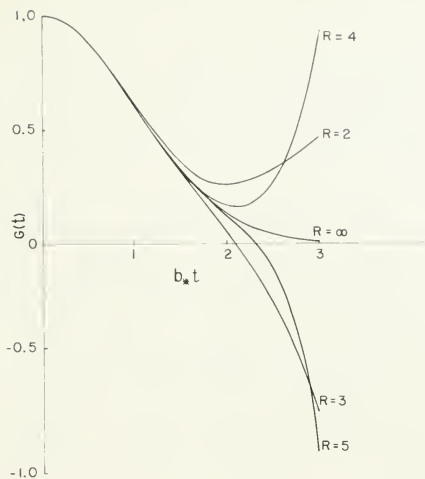


FIG. 5. SOLUTIONS OF (6.2) COMPARED WITH $G(t)$
FOR THE TRUE PROBLEM ($R=\infty$)

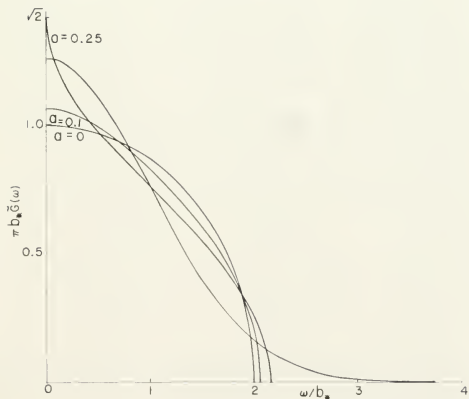


FIG. 6 SOLUTIONS $\tilde{G}(\omega)$ OF (6.6) FOR SEVERAL VALUES OF α
COMPARED WITH $\tilde{G}(\omega)$ FOR THE TRUE PROBLEM

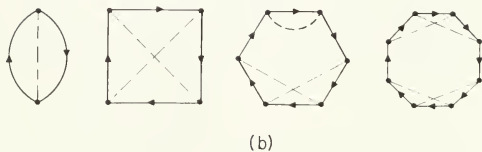
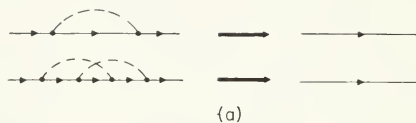


FIG. 7. (a) CONTRACTION OPERATIONS FOR (6.2) WITH $R=2$

(b) TYPICAL DIAGRAMS INCLUDED BY (6.2) WITH $R=2$

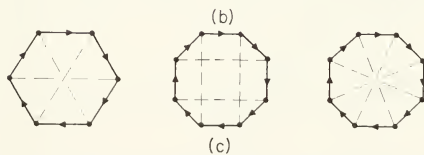
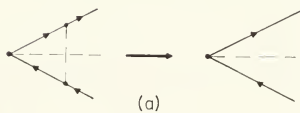


FIG. 8. (a) CONTRACTION OPERATION FOR THE SECOND STOCHASTIC MODEL

(b), (c) TYPICAL DIAGRAMS INCLUDED AND EXCLUDED IN THE SECOND STOCHASTIC MODEL

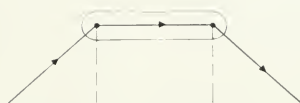


FIG. 9. FIXED DIAGRAM PART FOR $J(t)$

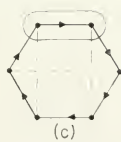
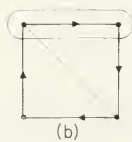
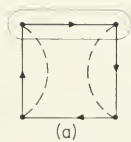


FIG. 10. THE SIMPLEST DIAGRAMS CONTRIBUTING TO $J(t)$

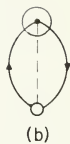
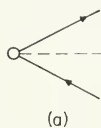


FIG. 11. (a) CONSOLIDATED VERTEX PART

(b) THE CONSOLIDATED DIAGRAM FOR $H(t)$

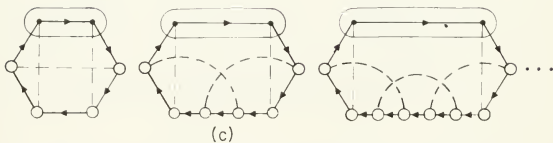
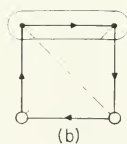
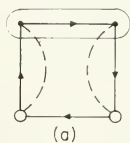


FIG. 12. CONSOLIDATED DIAGRAMS CONTRIBUTING TO $J(t)$ IN THE SECOND STOCHASTIC MODEL

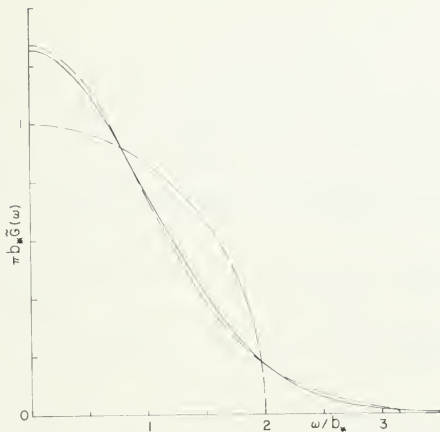


FIG. 13. COMPARISON OF $\tilde{G}(\omega)$ FOR THE RANDOM COUPLING MODEL AND THE SECOND STOCHASTIC MODEL (DASHED CURVES) AND FOR THE TRUE PROBLEM

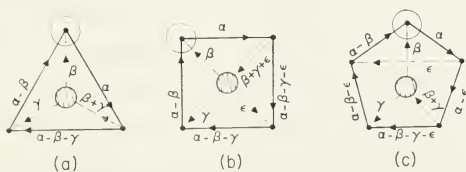


FIG. 14. DIAGRAMS FOR SIMPLE NON-GAUSSIAN CONTRIBUTIONS TO $G(t)$

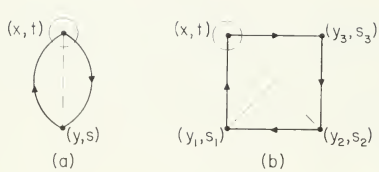


FIG. 15 DIAGRAMS FOR $\zeta_{2,1}(\underline{x}, t | \underline{x}', t')$ AND $\zeta_{4,3}(\underline{x}, t | \underline{x}', t')$

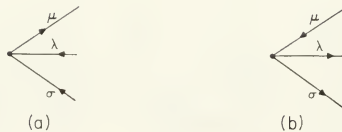


FIG. 16 VERTEX REPRESENTATIONS FOR THE TURBULENCE PROBLEM

FOOTNOTES

1. E. Montroll and J. Ward, Phys. Fluids 1, 55 (1958).
2. P.C. Martin and J. Schwinger, Phys. Rev. 115, 1342 (1959).
This paper contains an extensive bibliography.
3. G.K. Batchelor, The Theory of Homogeneous Turbulence (Cambridge University Press, Cambridge, England, 1953).
4. C.C. Lin, Turbulent Flows and Heat Transfer (Princeton University Press, Princeton, 1959), Chap. 1
5. J.B. Keller, in Hydrodynamic Instability and Related Problems, G. Birkhoff, editor (McGraw-Hill, New York, to be published).
6. See, for example: L. Van Hove, Physica 22, 343 (1956).
7. I. Proudman and W.H. Reid, Phil. Trans. Roy. Soc. (London) A 247, 163 (1954).
8. The reason for the peculiar bracket notation will become clear in Section 3.
9. Throughout this paper, quantities referring to individual systems will be labelled with square-bracketed Roman indices and collective quantities will be labelled with unbracketed Greek indices.
10. The special role played by $\alpha = 0$ will become clear in Section 8.
11. It is of interest to indicate how the q_α might be measured, in principle.
Let a device sample each oscillator in the collection in turn at time intervals τ , proceeding in order of increasing n and returning from the M th oscillator to the first to repeat the cycle continuously. At each sampling instant let the device produce a sharp pulse, of strength proportional to $q_{[n]}(t)$. If τ is small enough that many cycles are completed before the $q_{[n]}$ change appreciably, the spectrum of the pulse train will approximate a line spectrum with frequencies $\alpha/M\tau$ and amplitudes proportional to the $q_\alpha(t)$.

12. Reversal of the direction of the arrow on a dashed line labelled λ corresponds to the trivial notation change $\lambda \rightarrow -\lambda$ for the summed index λ in the associated $C_{2n;p}(\alpha, \beta, \alpha - \beta)$.

13. See Appendix A.

14. This model is unrelated to the 'random phase approximation' [D. Pines and D. Bohm, Phys. Rev. 85, 338 (1952)]. We make no assumption about the phases of the q_α themselves.

15. In general, $S_{2n} \neq (n-1)^{n-1}$.

16. The relations between irreducible and reducible C's, to which we have appealed in discussing (6.2), are not affected by the unrealizability of the C's. These relations may be regarded here as formal implications of the requirement that (4.14) agree with the power series for $G(t)$ obtained by the iteration solution of (4.23).

17. It is easy to see that any diagram which is transformable into the diagram for $C_{2;1}$ by contraction operations which involve the fixed vertex may also be so transformed by an alternative sequence of contractions which do not involve this vertex. Thus the diagrams included in this consolidated diagram are exhaustive.

18. The argument p in $\hat{G}(p)$, etc., should not be confused with the index p in $C_{2n;p}$.

19. See Appendix A.

20. Eq. (8.7) constitutes a very weak additional constraint on the ϕ 's, and we anticipate that, provided $C_{2;1} = 1$, it will not affect the realizability of any given set of values for the $C_{2n;p}$ when $M \rightarrow \infty$. This is verifiably so for the random coupling model: The sums $\sum_p C_{2n;p}(\alpha, \beta, \alpha - \beta)$ are unaffected ($M \rightarrow \infty$) for either $\alpha = 0$ or $\alpha \neq 0$.

21. The sums over ϕ products which occur in the present case involve $\sum'$ rather than the unrestricted summation by which the $C_{2n;p}$ are defined in Section 4. This does not alter the values of the sums in the limit $M \rightarrow \infty$, however.
22. Cumulant-discard approximations of the type discussed in Section 2 do not possess this consistency property and, in general, do not yield (8.29). The cumulant-discard approach is less satisfactory for determining $Q(t, t')$ than for determining $G(t)$. Cf. R.H. Kraichnan, Phys. Rev. 107, 1485 (1957) for a discussion of an analogous situation.
23. This property of (5.5) is directly implied by (2.4).
24. However, their number increases correspondingly so that those sums over individual skew moments which contribute to physical quantities remain finite in the limit.
25. We take units such that $\hbar = 1$ and $2m = 1$, where m is the particle mass.
26. The two problems may be regarded as formally identical if b and q are interpreted as vectors in a function space and a correspondence is established between d/dt and $\partial/\partial t - i\nabla^2$.
27. When the latter condition is not fulfilled, an additional term $i\bar{v}(\underline{x})G(\underline{x}, t | \underline{x}', t')$ appears on the left side of (10.4), where $\bar{v}(\underline{x})$ is the mean ^{then} potential, and $V(\underline{x}, \underline{y})_{\wedge}$ is defined in terms of the zero-mean part of the potential.
28. This procedure can be justified a posteriori.
29. $V_{\underline{k}}$ necessarily is real and non-negative.
30. We may note that $\int d^3k G_{\underline{k}}(-i\theta^{-1}) = G(\underline{x}, -i\theta^{-1} | \underline{x}, 0)$ is the partition function per unit volume, apart from normalization.
31. J. M. Burgers, Adv. Appl. Mech. 1, 171 (1948).
32. We have, in fact, assumed this condition in writing (11.4).

33. The additional symmetry properties expressed by (11.5) result in an ambiguity in the formal expressions for the $C_{2n;p}$ given by the rules in Section 4. There is, however, no ambiguity in value.
34. Only terms involving solely diagonal elements of the Green's function matrix are of leading order ($M \rightarrow \infty$). Thus, for example, the variation in u_a induced by the additional terms in the equation of motion for u_p does not contribute in the limit.
35. We use superscripts to label individual and collective quantities here in order to avoid possible confusion with tensor indices.
36. R.H. Kraichnan, J. Fluid Mech. 5, 497 (1959); Second Symposium on Naval Hydrodynamics, R. Cooper, editor (National Academy of Sciences-National Research Council, Washington, in press). The equations corresponding to the random coupling model are called the 'direct-interaction approximation' equations in these papers.
37. $G(\underline{x}, t | \underline{x}', t') d^3x$ is the probability that a marked particle introduced at $\underline{x}', t'$ is in d^3x at $\underline{x}, t$.
38. P.H. Roberts, to be published. [Issued also as report HSN-2, Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University, 1960.]
39. R.H. Kraichnan, in Hydrodynamic Instability and Related Problems, G. Birkhoff, editor (McGraw-Hill, New York, to be published).

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